



$C_m$  are the multiset  $\Delta(C_m) = \{\pm|c_i - c_{i-1}| \mid i = 1, 2, \dots, m\}$  where  $c_0 = c_m$ . Let  $\mathcal{F} = \{B_1, B_2, \dots, B_r\}$  be an  $m$ -cycles of  $\lambda K_v$ , the list of differences from  $\mathcal{F}$  is  $\Delta(\mathcal{F}) = \bigcup_{i=1}^r \Delta(B_i)$ .

The orbit of cycle  $C_m$ , denoted by  $orb(C_m)$ , is the set of all distinct  $m$ -cycles in the collection  $\{C_m + i \mid i \in Z_v\}$ . The length of  $orb(C_m)$  is its cardinality, i.e.,  $orb(C_m) = k$  where  $k$  is the minimum positive integer such that  $C_m + k = C_m$ . A cycle orbit of length  $v$  on  $\lambda K_v$  is said full and otherwise short. [10]

The stabilizer of a subgraph  $H$  of a graph  $G$  of order  $v$  is  $stab(H) = \{z \in Z_v \mid z + H = H\}$  and  $H$  has trivial stabilizer when  $stab(H) = \{0\}$ . One may easily deduce the following result.

For presenting a cyclic  $m$ -cycle system of  $\lambda K_v$ , it sufficient to construct a starter set, i.e.,  $m$ -cycle system of representations for its cycle orbits. As particular consequences of the theory developed in [11] we have:

Lemma 1. Let  $H$  be a subgraph of  $G$  and  $|stab(H)| > 1$ . Then each nonzero integer in  $\Delta H$  appears a multiple of  $|stab(H)|$  times.

Lemma 2. Let  $\delta$  be a multiset of subgraphs of  $\lambda K_v$  and every subgraph of  $\delta$  has trivial stabilizer. Then  $\delta$  is a starter of cyclic  $(\lambda K_v, \mathcal{Y})$ -design if and only if  $\Delta\delta$  covers each nonzero integer of  $Z_v$  exactly  $\lambda$  times.

### III. CYCLIC NEAR HAMILTONIAN CYCLE SYSTEM

Definition 1. A full cyclic near Hamiltonian cycle system of the  $4K_v$ , denoted by  $CNHC(4K_v, C_{v-1})$ , is a cyclic  $(v-1)$ -cycle system of  $4K_v$  graph, that satisfies the following conditions:

1. The cycle in row  $r$  form a near-4-factor with focus  $r$ .
2. The cycles associated with the rows contain no repetitions.

Surely, for presenting a full cyclic near Hamiltonian cycle system of the  $4K_v$ ,  $CNHC(4K_v, C_{v-1})$ , it is sufficient to provide a set of starter set that satisfies a near-4-factor. We give here example to explain the above definition.

Example 1. Let  $G = 4K_{14}$  and  $\mathcal{F} = \{C_{13}, C_{13}^*\}$  is a set of 13-cycles of  $G$  such that:

$$\begin{aligned} C_{13} &= (1, 13, 2, 12, 3, 11, 4, 5, 10, 6, 9, 7, 8), \\ C_{13}^* &= (13, 8, 12, 9, 11, 10, 4, 3, 5, 2, 6, 1, 7). \end{aligned}$$

Firstly, it is easy to observe that each non zero element in  $Z_{14}$  occurs exactly twice in the 13-cycles of  $\mathcal{F}$ . Since, the cycle graph is 2-regular graph, then every vertex has a degree 4 except a zero element (isolated vertex) has a degree zero. Thus, it is satisfies the near-4-factor with focus zero element. Secondly, the list of differences set of the set  $\mathcal{F}$  is listed in Table I

TABLE I  
THE LIST OF DIFFERENCES OF  $\mathcal{F}$

13-cycles	Difference set
(1,13,2,12,3,11,4,5,10,6,9,7,8)	$\{2,12,3,11,4,10,5,9,6,8,7,7,1,13,\}$ $\{5,9,4,10,3,11,2,12,1,13,7,7\}$
(13,8,12,9,11,10,4,3,5,2,6,1,7)	$\{5,9,4,10,3,11,2,12,1,13,6,8,1,13,\}$ $\{2,12,3,11,4,10,5,9,6,8,6,8\}$

It can be seen from the Table I,  $\Delta(\mathcal{F}) = \Delta(C_{13}) \cup \Delta(C_{13}^*)$  covers each nonzero element in  $Z_{14}$  exactly four times. Since the cycles set  $\mathcal{F}$  has trivial stabilizer based on Lemma 1, then the set  $\mathcal{F} = \{C_{13}, C_{13}^*\}$  is the starter set of  $CNHC(4K_{14}, C_{13})$  by Lemma 2.

Therefore,  $CNHC(4K_{14}, C_{13})$  is an  $(14 \times 2)$  array design and cycles set  $\mathcal{F} = \{C_{13}, C_{13}^*\}$  in the first row generates all cycles in  $(14 \times 2)$  array by repeated addition of 1 modular 14 as shown in the Table II.

TABLE II  
 $CNHC(4K_{14}, C_{13})$

Focus	$CNHC(4K_{14}, C_{13})$	
$r = 0$	(1,13,2,12,3,11,4,5,10,6,9,7,8)	(13,8,12,9,11,10,4,3,5,2,6,1,7)
$r = 1$	(2,0,3,13,4,12,5,6,11,7,10,8,9)	(0,9,13,10,12,11,5,4,6,3,7,2,8)
$r = 2$	(3,1,4,0,5,13,6,7,12,8,11,9,10)	(1,10,0,11,13,12,6,5,7,4,8,3,9)
$\vdots$	$\vdots$	$\vdots$
$r = 13$	(0,12,1,1,2,10,3,4,9,5,8,6,7)	(12,7,11,8,10,9,3,2,4,1,5,0,6)

Throughout the paper, a near Hamiltonian cycle of order  $(v-1)$  will be represented as connected paths, we mean that  $C_{v-1} = (c_{(1,1)}, P_{(1,2)}^{2n}, P_{(3,4)}^{2n})$  where,  $P_{(1,2)}^{2n}$  and  $P_{(3,4)}^{2n}$  are  $(2n)$ -paths such that:

$$\begin{aligned} P_{(1,2)}^{2n} &= [c_{(1,1)}, c_{(2,1)}, c_{(1,2)}, c_{(2,2)}, \dots, c_{(1,n)}, c_{(2,n)}], \\ &= [\bigcup_{i=1}^n c_{(1,i)}, c_{(2,i)}] \\ P_{(3,4)}^{2n} &= [c_{(3,1)}, c_{(4,1)}, c_{(3,2)}, c_{(4,2)}, \dots, c_{(3,n)}, c_{(4,n)}] \\ &= [\bigcup_{i=1}^n c_{(3,i)}, c_{(4,i)}]. \end{aligned}$$

Let the vertex sets of  $P_{(1,2)}^{2n}$  and  $P_{(3,4)}^{2n}$  are  $\{\bigcup_{i=1}^n c_{(1,i)}, \bigcup_{i=1}^n c_{(2,i)}\}, \{\bigcup_{i=1}^n c_{(3,i)}, \bigcup_{i=1}^n c_{(4,i)}\}$ , respectively. And the list of difference sets of  $P_{(1,2)}^{2n}$  and  $P_{(3,4)}^{2n}$  will be calculated as follows:

$$\begin{aligned} \Delta(P_{(1,2)}^{2n}) &= \Delta_1(P_{(1,2)}^{2n}) \cup \Delta_2(P_{(1,2)}^{2n}), \\ \Delta(P_{(3,4)}^{2n}) &= \Delta_1(P_{(3,4)}^{2n}) \cup \Delta_2(P_{(3,4)}^{2n}) \text{ such that} \\ \Delta_1(P_{(1,2)}^{2n}) &= \{\pm|c_{(1,i)} - c_{(2,i)}| \mid 1 \leq i \leq n\}. \\ \Delta_2(P_{(1,2)}^{2n}) &= \{\pm|c_{(1,i+1)} - c_{(2,i)}| \mid 1 \leq i \leq n-1\}. \\ \Delta_1(P_{(3,4)}^{2n}) &= \{\pm|c_{(3,i)} - c_{(4,i)}| \mid 1 \leq i \leq n\}. \\ \Delta_2(P_{(3,4)}^{2n}) &= \{\pm|c_{(3,i+1)} - c_{(4,i)}| \mid 1 \leq i \leq n-1\}. \end{aligned}$$

And we define  $\Delta(c_{(1)}, P_{(1,2)}^{2n}), \Delta(P_{(3,4)}^{2n}, c_{(1)})$  and  $\Delta(P_{(1,2)}^{2n}, P_{(3,4)}^{2n})$  as follows

$$\begin{aligned}\Delta(c_{(1)}, P_{(1,2)}^{2n}) &= \pm |c_{(1)} - c_{(1,1)}| \\ \Delta(P_{(3,4)}^{2n}, c_{(1)}) &= \pm |c_{(4,n)} - c_{(1)}| \\ \Delta(P_{(1,2)}^{2n}, P_{(3,4)}^{2n}) &= \pm |c_{(2,n)} - c_{(3,1)}|.\end{aligned}$$

So, the list of difference of  $C_{v-1}$  shall be represented as follows:

$$\begin{aligned}\Delta(C_{v-1}) &= \Delta(P_{(1,2)}^{2n}) \cup \Delta(P_{(3,4)}^{2n}) \cup \Delta(c_{(1)}, P_{(1,2)}^{2n}) \cup \\ &\Delta(P_{(3,4)}^{2n}, c_{(1)}) \cup \Delta(P_{(1,2)}^{2n}, P_{(3,4)}^{2n})\end{aligned}$$

Now we are able to provide our main result.

**Theorem 1.** There exists a full cyclic near Hamiltonian cycle system of  $4K_v$ ,  $CNHC(4K_v, C_{v-1})$ , when  $v = 4n + 2, n > 2$ .

**Proof.** Suppose  $\mathcal{F} = \{C_{4n+1}, C_{4n+1}^*\}$  is a set of near Hamiltonian cycles of  $4K_{4n+2}$  where

$$\begin{aligned}C_{4n+1} &= (1, P_{(1,2)}^{2n}, P_{(3,4)}^{2n}), \\ C_{4n+1}^* &= (2n + 1, P_{(1,2)}^{(2n)*}, P_{(3,4)}^{(2n)*})\end{aligned}$$

Such that:

- $P_{(1,2)}^{2n} = [4n + 1, 2, 4n, 3, \dots, 3n + 2, n + 1]$   
 $= [\cup_{i=1}^n 4n + 2 - i, i + 1]$
- $P_{(3,4)}^{2n} = [n + 2, 3n + 1, n + 3, 3n, \dots, 2n + 1, 2n + 2]$   
 $= [\cup_{i=1}^n n + 1 + i, 3n + 2 - i]$
- $P_{(1,2)}^{(2n)*} = [4n + 1, 2n + 2, 4n, 2n + 3, \dots, 3n + 2, 3n + 1] = [\cup_{i=1}^n 4n + 2 - i, 2n + 1 + i]$
- $P_{(3,4)}^{(2n)*} = [n + 1, n, n + 2, n - 1, \dots, 2n, 1]$   
 $= [\cup_{i=1}^n n + i, n + 1 - i].$

We will divide the proof into two parts as follows:

**Part 1.** In this part will be proved that  $\mathcal{F}$  satisfies a near-4-factor. We shall calculate the vertex set of  $C_{4n+1}$  and  $C_{4n+1}^*$  such that:

$$V(C_{4n+1}) = V(P_{(1,2)}^{2n}) \cup V(P_{(3,4)}^{2n}) \cup \{1\}.$$

$$V(C_{4n+1}^*) = V(P_{(1,2)}^{(2n)*}) \cup V(P_{(3,4)}^{(2n)*}) \cup \{2n + 1\}.$$

$$\begin{aligned}\cup_{i=1}^n c_{(1,i)} &= \{4n + 2 - i, 1 \leq i \leq n\} \\ &= \{4n + 1, 4n, \dots, 3n + 2\},\end{aligned}\quad (1)$$

$$\cup_{i=1}^n c_{(2,i)} = \{i + 1, 1 \leq i \leq n\} = \{2, 3, \dots, n + 1\}, \quad (2)$$

$$\begin{aligned}\cup_{i=1}^n c_{(3,i)} &= \{n + 1 + i, 1 \leq i \leq n\} \\ &= \{n + 2, n + 3, \dots, 2n + 1\},\end{aligned}\quad (3)$$

$$\begin{aligned}\cup_{i=1}^n c_{(4,i)} &= \{3n + 2 - i, 1 \leq i \leq n\} \\ &= \{3n + 1, 3n, \dots, 2n + 2\}.\end{aligned}\quad (4)$$

From above equations, it is easy to notice that  $V(C_{4n+1})$  covers each nonzero element of  $Z_{4n+2}$  exactly once.

$$\begin{aligned}\cup_{i=1}^n c_{(1,i)}^* &= \{4n + 2 - i, 1 \leq i \leq n\} \\ &= \{4n + 1, 4n, \dots, 3n + 2\},\end{aligned}\quad (5)$$

$$\begin{aligned}\cup_{i=1}^n c_{(2,i)}^* &= \{2n + 1 + i, 1 \leq i \leq n\} \\ &= \{2n + 2, 2n + 3, \dots, 3n + 1\},\end{aligned}\quad (6)$$

$$\begin{aligned}\cup_{i=1}^n c_{(3,i)}^* &= \{n + i, 1 \leq i \leq n\} \\ &= \{n + 1, n + 2, \dots, 2n\}\end{aligned}\quad (7)$$

$$\cup_{i=1}^n c_{(4,i)}^* = \{n + 1 - i\} = \{n, n - 1, \dots, 1\} \quad (8)$$

It can be observed from the above equations that  $V(C_{4n+1}) = V(C_{4n+1}^*)$ . Then, the multiset  $V(C_{4n+1}) \cup V(C_{4n+1}^*)$  covers each nonzero elements of  $Z_{4n+2}$  exactly twice. Since the cycle graph is 2-regular graph, therefore  $\mathcal{F} = \{C_{4n+1}, C_{4n+1}^*\}$  satisfies a near-four-factor (with focus zero).

**Part 2.** In this part we will prove  $\mathcal{F} = \{C_{4n+1}, C_{4n+1}^*\}$  is the starter set of cyclic  $(v - 1)$ -cycle system of  $4K_v$ . So, we will calculate the difference set of each of them as follows:

$$\begin{aligned}\Delta(C_{4n+1}) &= \Delta(P_{(1,2)}^{2n}) \cup \Delta(P_{(3,4)}^{2n}) \cup \Delta(c_{(1)}, P_{(1,2)}^{2n}) \cup \\ &\Delta(P_{(1,2)}^{2n}, P_{(3,4)}^{2n}) \cup \Delta(P_{(3,4)}^{2n}, c_{(1)})\end{aligned}\quad (9)$$

$$\begin{aligned}\Delta_1(P_{(1,2)}^{2n}) &= \cup_{i=1}^n \pm |4n + 1 - 2i| = \{4n - 1, 4n - 3, \dots, 2n + 1\} \cup \{3, 5, \dots, 2n + 1\}\end{aligned}$$

$$\begin{aligned}\Delta_2(P_{(1,2)}^{2n}) &= \cup_{i=1}^{n-1} \pm |4n - 2i| = \{4n - 2, 4n - 4, \dots, 2n + 2\} \cup \{4, 6, \dots, 2n\}\end{aligned}$$

$$\begin{aligned}\Delta_1(P_{(3,4)}^{2n}) &= \cup_{i=1}^n \pm |2n + 1 - 2i| = \{2n - 1, 2n - 3, \dots, 1\} \cup \{2n + 3, 2n + 5, \dots, 4n + 1\}\end{aligned}$$

$$\begin{aligned}\Delta_2(P_{(3,4)}^{2n}) &= \cup_{i=1}^{n-1} \pm |2n - 2i| = \{2n - 2, 2n - 4, \dots, 2\} \cup \{2n + 4, 2n + 6, \dots, 4n\}\end{aligned}$$

$$\Delta(c_{(1)}, P_{(1,2)}^{2n}) = \pm |c_{(1)} - c_{(1,1)}| = \{2, 4n\},$$

$$\Delta(P_{(1,2)}^{2n}, P_{(3,4)}^{2n}) = \pm |c_{(2,n)} - c_{(3,1)}| = \{1, 4n + 1\}.$$

$$\Delta(P_{(3,4)}^{2n}, c_{(1)}) = \pm |c_{(4,n)} - c_{(1)}| = \{2n + 1, 2n + 1\}$$

From the equation 9, We note that the list of differences of  $C_{4n+1}$ ,  $\Delta(C_{4n+1})$ , covers each nonzero elements of  $Z_{4n+2}$  twice except the differences  $\{2n, 2n + 2\}$  appear once.

Now we will calculate  $\Delta(C_{4n+1}^*)$  such as

$$\begin{aligned}\Delta(C_{4n+1}^*) &= \Delta(P_{(1,2)}^{(2n)*}) \cup \Delta(P_{(3,4)}^{(2n)*}) \cup \Delta(c_{(1)}, P_{(1,2)}^{(2n)*}) \cup \\ &\Delta(P_{(1,2)}^{(2n)*}, P_{(3,4)}^{(2n)*}) \cup \Delta(P_{(3,4)}^{(2n)*}, c_{(1)}^*)\end{aligned}\quad (10)$$

$$\begin{aligned}\Delta_1(P_{(1,2)}^{(2n)*}) &= \cup_{i=1}^n \pm |2n + 1 - 2i| = \{2n - 1, 2n - 3, \dots, 1\} \cup \{2n + 3, 2n + 5, \dots, 4n + 1\}\end{aligned}$$

- $\Delta_2(P_{(1,2)}^{(2n)^*}) = \cup_{i=1}^{n-1} \pm |2n - 2i| = \{2n - 2, 2n - 4, \dots, 2\} \cup \{2n + 4, 2n + 6, \dots, 4n\}$
- $\Delta_1(P_{(3,4)}^{(2n)^*}) = \cup_{i=1}^n \pm |2i - 1| = \{1, 3, \dots, 2n - 1\} \cup \{4n + 1, 4n - 1, \dots, 2n + 3\}$
- $\Delta_2(P_{(3,4)}^{(2n)^*}) = \cup_{i=1}^{n-1} \pm |2i| = \{2, 4, \dots, 2n - 2\} \cup \{4n, 4n - 2, \dots, 2n + 4\}$
- $\Delta(c_{(1),P_{(1,2)}^{(2n)^*}}^*) = \pm |c_{(1)}^* - c_{(1,1)}^*| = \{2n, 2n + 2\}$ .
- $\Delta(P_{(1,2)}^{(2n)^*}, P_{(3,4)}^{(2n)^*}) = \pm |c_{(2,n)}^* - c_{(3,1)}^*| = \{2n, 2n + 2\}$
- $\Delta(P_{(3,4)}^{(2n)^*}, c_{(1)}^*) = \pm |c_{(4,n)}^* - c_{(1)}^*| = \{2n, 2n + 2\}$ .

As clearly shown, in the equations 10, every nonzero element in  $Z_{4n+2}$  appears twice except  $\{2n, 2n + 2\}$  appear three times in  $\Delta(C_{4n+1}^*)$ . Based on Lemma 1, the cycles  $\{C_{4n+1}, C_{4n+1}^*\}$  have trivial stabilizer.

One can easily note that  $\Delta(\mathcal{F}) = \Delta(C_{4n+1}) \cup \Delta(C_{4n+1}^*)$  covers each non zero integers in  $Z_{4n+2}$  four times. Thus,  $\mathcal{F} = \{C_{4n+1}, C_{4n+1}^*\}$  is the starter cycles of cyclic  $(v - 1)$ -cycle system of  $4K_v$  by Lemma 2. Hence, the cycles set  $\mathcal{F} = \{C_{4n+1}, C_{4n+1}^*\}$  generates a full near Hamiltonian cycle system of  $4K_v$  by adding one modular  $v$  when  $v = 4n + 2, n > 2$

#### IV. CYCLIC HAMILTONIAN WHEEL SYSTEM

A wheel graph of order  $m$ , denoted by  $W_m$ , consists of a singleton graph  $K_1$  and a cycle graph of order  $m - 1, C_{m-1}$ , in which the  $K_1$  is connected to all the vertices of  $C_{m-1}$ , written  $K_1 + C_{m-1}$  or  $c_0 + (c_1, c_2, \dots, c_{m-1})$ . An  $m$ -wheel contains  $2(m - 1)$  edges such that the edge set of  $W_m$  is  $E(W_m) = E(K_{1,(m-1)}) \cup E(C_{m-1})$  [12].

An  $m$ -wheel system of graph  $G$  is a decomposition of edge set of  $G$  into collection  $\mathcal{W} = \{W_{m_1}, \dots, W_{m_r}\}$  of edges-disjoint of  $m$ -wheels. Similar to the cyclic cycle system, an  $m$ -wheel system of  $\lambda K_v$  is a cyclic if  $V(\lambda K_v) = Z_v$  and if  $W_m = c_0 + (c_1, c_2, \dots, c_{m-1}) \in \mathcal{W}$  implies that  $W_m + 1 = (c_0 + 1) + (c_1 + 1, c_2 + 1, \dots, c_{m-1} + 1)$  is also in  $\mathcal{W}$ . Moreover, if  $m = v$  then it is called a cyclic Hamiltonian wheel system. The list of difference of  $W_m = c_0 + (c_1, c_2, \dots, c_{m-1})$  is  $\Delta(W_m) = \Delta(K_{1,(m-1)}) \cup \Delta(C_{m-1})$  such that  $\Delta(C_m) = \{\pm |c_{i+1} - c_i| \mid 1 \leq i \leq m - 1\}$  where  $c_m = c_0$  and  $\Delta(K_{1,(m-1)}) = \{\pm |c_i - c_0| \mid 1 \leq i \leq m - 1\}$ . More generally, given a multiset  $\mathcal{W} = \{W_{m_1}, \dots, W_{m_r}\}$  of  $m$ -wheels of  $\lambda K_v$ , the list of differences from  $\mathcal{W}$  is  $\Delta \mathcal{W} = \cup_{i=1}^r \Delta W_{m_i}$ .

Definition 2. The full cyclic Hamiltonian wheel system of a graph  $8K_v$ , denoted by  $CHWS(8K_v, \mathcal{W})$ , is a cyclic  $(v)$ -wheel system of a graph  $8K_v$  that generated by starter set  $\mathcal{W} = \{W_{v_1}, \dots, W_{v_r}\}$  such that the associated cycles with wheels satisfy near-four-factor with focus singleton graph.

In other words  $CHWS(8K_v, \mathcal{W})$  is a  $(v \times |\mathcal{W}|)$  array such that satisfies the following conditions:

- 1) The wheels in row  $r$  form a  $r$  + (near-four-factor).
- 2) The wheels associated with the rows contain no repetitions.

For clarity, we provide an example to demonstrate the construction of a cyclic Hamiltonian wheel system stated above.

Example 2. Let  $G = 8K_{14}$  and  $\mathcal{W} = \{W_{14}, W_{14}^*\}$  is a set of Hamiltonian wheels of  $8K_{14}$ .

Where

$$W_{14} = K_1 + C_{13} = 0 + (1, 13, 2, 12, 3, 11, 4, 5, 10, 6, 9, 7, 8)$$

and

$$W_{14}^* = K_1^* + C_{13}^* = 0 + (13, 8, 12, 9, 11, 10, 4, 3, 5, 2, 6, 1, 7)$$

From the Example 1, we can note that the 13-cycles  $\{C_{13}, C_{13}^*\}$  satisfy a near-four-factor with focus 0 (zero element). Moreover, the list of differences from  $\{C_{13}, C_{13}^*\}$  covers every nonzero element of  $Z_{14}$  exactly four times. Now we want to find the list of differences from  $(K_{1,(13)}) \cup (K_{1,(13)}^*)$  as a follows

$$\begin{aligned} \Delta(K_{1,(13)}) &= \{\pm |c_i - c_0| \mid 1 \leq i \leq 13\} \\ &= \{\pm |c_i| \mid 1 \leq i \leq 13\} \end{aligned}$$

such that the  $c_i \in V(C_{13}), 1 \leq i \leq 13$ . Since the vertex set of  $C_{13}$  is  $Z_{14} - \{0\}$  and  $Z_{14} = -Z_{14}$ , then  $\Delta(K_{1,(13)})$  covers each nonzero element of  $Z_{14}$  twice. Similarly, one can be found  $\Delta(K_{1,(13)}^*) = \Delta(K_{1,(13)})$ . So  $\Delta(\mathcal{W})$  covers each nonzero of  $Z_{14}$  four times. Thus,  $\mathcal{W} = \{W_{14}, W_{14}^*\}$  is the starter set of  $CHWS(8K_{14}, \mathcal{W})$ .

Then  $CHWS(8K_{14}, W_{14})$  is an  $(14 \times 2)$  array deign where all its wheels can be generated by repeated addition 1 (modular 14) on the starter set  $\mathcal{W}$  as shown in the Table III.

TABLE III  
 $CHWS(8K_{14}, W_{14})$

$CHWS(8K_{14}, \mathcal{W})$	
$0 + (1,13,2,12,3,11,4,5,10,6,9,7,8)$	$0 + (13,8,12,9,11,10,4,3,5,2,6,1,7)$
$1 + (2,0,3,13,4,12,5,6,11,7,10,8,9)$	$1 + (0,9,13,10,12,11,5,4,6,3,7,2,8)$
$2 + (3,1,4,0,5,13,6,7,12,8,11,9,10)$	$2 + (1,10,0,11,13,12,6,5,7,4,8,3,9)$
$\vdots$	$\vdots$
$13 + (0,12,1,1,2,10,3,4,9,5,8,6,7)$	$13 + (12,7,11,8,10,9,3,2,4,1,5,0,6)$

The following theorem proves the existence of  $CHWS(8K_{4n+2}, \mathcal{W})$ .

Theorem 2. There exists a full cyclic Hamiltonian wheel system of  $8K_v, CHWS(8K_v, \mathcal{W})$ , for  $v = 4n + 2, n > 2$ .

Proof. We have to present a starter set  $\mathcal{W} = \{K_1 + C_{4n+1}, K_1^* + C_{4n+1}^*\}$  of  $CHWS(8K_v, \mathcal{W})$  such

that the cycles associated with the wheels in  $\mathcal{W}$  satisfy a near-four-factor with focus a singleton graph.

Suppose  $\mathcal{W} = \{0 + C_{4n+1}, 0 + C_{4n+1}^*\}$  is a set of Hamiltonian wheels of  $8K_{4n+2}$  where

$$C_{4n+1} = (1, P_{(1,2)}^{2n}, P_{(3,4)}^{2n}),$$

$$C_{4n+1}^* = (2n+1, P_{(1,2)}^{(2n)*}, P_{(3,4)}^{(2n)*})$$

Such that:

- $P_{(1,2)}^{2n} = [4n+1, 2, 4n, 3, \dots, 3n+2, n+1].$
- $P_{(3,4)}^{2n} = [n+2, 3n+1, n+3, 3n, \dots, 2n+1, 2n+2].$
- $P_{(1,2)}^{(2n)*} = [4n+1, 2n+2, 4n, 2n+3, \dots, 3n+2, 3n+1].$
- $P_{(3,4)}^{(2n)*} = [n+1, n, n+2, n-1, \dots, 2n, 1].$

From Theorem 1, the cycles associated with the Hamiltonian wheels in  $\mathcal{W}$  satisfy the near-four-factor with focus zero element.

Now, we want to prove  $\mathcal{W} = \{K_1 + C_{4n+1}, K_1^* + C_{4n+1}^*\}$  is a  $(\lambda K_v, \mathcal{W})$ -difference system. To do this, it is enough to show that the list of differences

$$\Delta\mathcal{W} = \{\Delta(C_{4n+1}) \cup \Delta(C_{4n+1}^*) \cup \Delta(K_{1,(4n+1)}^*) \cup \Delta(K_{1,(4n+1)})\}$$

covers each element of  $\{Z_{4n+2} - \{0\}\}$  eight times. Firstly, as indicated in Theorem 1, the list of differences of  $\{(C_{4n+1}) \cup (C_{4n+1}^*)\}$  cover each nonzero element in  $Z_{4n+2}$  exactly four times.

Secondly, the list of differences of  $(K_{1,(4n+1)})$  is  $\{\pm |c_i - 0| \mid c_i \in C_{4n+1}\}$ . Since  $V(C_{4n+1}) = Z_{4n+2} - \{0\}$  then  $\{|c_i - 0| \mid c_i \in C_{4n+1}\} = Z_{4n+2} - \{0\}$ . Because of  $Z_{4n+2} = -\{Z_{4n+2}\}$ , then  $\Delta(K_{1,(4n+1)}) = \{\pm |c_i - 0| \mid c_i \in C_{4n+1}\}$  covers each nonzero element of  $Z_{4n+2}$  twice. Likewise, we repeat the same strategy on cycle  $K_{1,(4n+1)}^*$  to find  $\Delta(K_{1,(4n+1)}^*)$ . Also, it is an easy matter to check that  $\Delta(K_{1,(4n+1)}^*) = \Delta(K_{1,(4n+1)})$ .

Linking together the above list of differences, we see that  $\Delta\mathcal{W}$  covers each nonzero element of  $Z_{4n+2}$  eight times. On the other hand, each wheel graph in  $\mathcal{W}$  has trivial stabilizer based on Lemma 1. Therefore,  $\mathcal{W}$  is the starter set of

$CHWS(8K_v, \mathcal{W})$ , by Lemma 2. One can be generated  $CHWS(8K_v, \mathcal{W})$  by repeated addition 1 modular  $v$  on  $\mathcal{W}$ .  $\square$

## V. CONCLUSION

In this paper, we have provided new designs  $CNHC(4K_v, C_{v-1})$  and  $CHWS(8K_v, W_v)$  where  $v \equiv 2 \pmod{4}$ . These designs are interested in a decomposition of complete multigraph into cyclic  $(v-1)$ -cycle and cyclic  $(v)$ -wheel graphs, respectively. We have also proved the existence of these designs by constructed the starter set for each of them. Moreover, one can ask if  $CNHC(2\lambda K_v, C_{v-1})$  and  $CHWS(2\lambda K_v, W_v)$  can be constructed for the case  $v \equiv 2, 4 \pmod{4}$  and  $\lambda > 2$

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