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Decompositions of Complete Multigraphs into Cyclic Designs

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***Abstract*— Let v and λ be positive integer, λK_v denote a complete multigraph. A decomposition of a graph G is a set of subgraphs of G whose edge sets partition the edge set of G. The aim of this paper, is to decompose a complete multigraph $4K_v$ into cyclic $(v-1)$-cycle system according to specified conditions. As the main consequence, construction of decomposition of $8K_v$ into cyclic Hamiltonian wheel system, where $v \equiv 2(mod\ 4)$, is also given. The difference set method is used to construct the desired designs.**

***Keywords*— Cyclic design; Hamiltonian cycle, Near four factor, Wheel graph.**

I. INTRODUCTION

Throughout this paper, all graphs consider finite and undirected. A complete graph of order v denotes by K_v. An (G,Y)-design is a decomposition of the graph G into subgraphs belonging to an assigned multiset Y.

An m-cycle, written C_m=(c_0,⋯,c_(m-1)), consists of m distinct vertices {c_0,c_1,⋯,c_(m-1) } and m edges {c_i c_(i+1) },0⩽i ⩽m-2 and c_0 c_(m-1) and m-cycle of a graph G is called Hamiltonian when its vertices passes through all the vertex set of G. An m-path, written [c_0, ⋯ ,c_(m-1)], consists of m distinct vertices {c_0,c_1,⋯,c_(m-1) } and m-1 edges {c_i c_(i+1) },0⩽i ⩽ m-2. An m-cycle system of a graph G is (G,C)-design where C is a collection of m-cycles. If G=K_v then such m-cycle system is called m-cycle system of order v and is also said a simple when its cycles are all distinct.

An automorphism group on (G,Y)-design is a bijections on V(G) fixed Y. An (G,Y)-design is a cyclic if it admit automorphism group acting regularly on V(G) [1]. For a cyclic (G,Y)-design, we can assume that V(G)=Z_v. So, the automorphism can be represented by

α :i→i+1 ("mod" v) or α :(0,1,⋯,v-1)

A starter set of a cyclic (G,Y)-design is a set of subgraphs of G that generates all subgraphs of Y by repeated addition of 1 modular v.

A complete multigraph of order v, denoted by λK_v, is obtained by replacing each edge of K_v with λ edges. The problem which concerned in the decomposition of the complete multigraph into subgraphs has received much attention in recent years. The necessary and sufficient conditions for decomposing λK_v into cycles of order λ and cycles of prime oredr have been established by [2]. While, the existence theorem of m-cycle system of λK_v has been proved for all values of λ in [3]. For the important case of λ=1, the existence question for m-cycle system of order v has been completely settled by [4] in the case m odd and by [5] in the case m even. Moreover, the cyclic m-cycle system of order v for m=3, denoted by CTS(v,λ), has been constructed by [6] and for a cyclic Hamiltonian cycle system of order v was proved when v is an odd integer but v≠15 and v≠p^α with p a prime and α >1 [7].

On the other hand, the necessary and sufficient conditions for decomposing λK_v into cycle and star graphs have been investigated by [8].

A four-factor of a graph G is a spanning subgraph whose vertices have a degree 4. While a near-four-factor is a spanning subgraph in which all vertices have a degree four with exception of one vertex (isolated vertex) which has a degree zero [9].

In this paper, we propose a new type of cyclic cycle system that is called cyclic near Hamiltonian cycle system of 4K_v, denoted CNHC(4K_v,C_(v-1)). This is obtained by combination a near-four-factors and cyclic (v-1)-cycle system of 4K_v when v≡2(mod 4). Furthermore, the construction of CNHC(4K_v,C_(v-1)) will be employed to decompose 8K_v into Hamiltonian wheels.

II. PRELIMINARIES

In our paper, all graphs considered have vertices in Z_v. We will use the difference set method to construct the desired designs. The difference between any two distinct vertices a and b in λK_v is $\pm|a-b|$, arithmetic $(mod\ v)$. Given $C_m=(c_0,\dots,c_{m-1})$ an m-cycle, the differences from

C_m are the multiset $\Delta(C_m) = \{\pm|c_i - c_{i-1}| \;|i = 1, 2, \dots, m\}$ where $c_0 = c_m$. Let $\mathcal{F} = \{B_1, B_2, \dots, B_r\}$ be an m-cycles of λK_v the list of differences from $\mathcal{F}$ is $\Delta(\mathcal{F}) = \cup_{i=1}^{r}\Delta(B_i)$.

The orbit of cycle C_m, denoted by $orb(C_m)$, is the set of all distinct m-cycles in the collection $\{C_m + i| \; i \in Z_v\}$. The length of $orb(C_m)$ is its cardinality, i.e., $orb(C_m) = k$ where k is the minimum positive integer such that $C_m + k = C_m$. A cycle orbit of length v on λK_v is said full and otherwise short. [10]

The stabilizer of a subgraph H of a graph G of order v is $stab(H) = \{z \in Z_v \mid z + H = H\}$ and H has trivial stabilizer when $stab(H) = \{0\}$. One may easily deduce the following result.

For presenting a cyclic m-cycle system of λK_v, it sufficient to construct a starter set, i.e., m-cycle system of representations for its cycle orbits. As particular consequences of the theory developed in [11] we have:

Lemma 1. Let H be a subgraph of G and $|stab(H)| > 1$. Then each nonzero integer in ΔH appears a multiple of $|stab(H)|$ times.

Lemma 2. Let δ be a multiset of subgraphs of λK_v and every subgraph of δ has trivial stabilizer. Then δ is a starter of cyclic $(\lambda K_v, \mathcal{Y})$-design if and only if $\Delta\delta$ covers each nonzero integer of Z_v exactly λ times.

III. Cyclic Near Hamiltonian Cycle System

Definition 1. A full cyclic near Hamiltonian cycle system of the $4K_v$, denoted by $CNHC(4K_v, C_{v-1})$, is a cyclic $(v-1)$-cycle system of $4K_v$ graph, that satisfies the following conditions:

1. The cycle in row r form a near-4-factor with focus r.
2. The cycles associated with the rows contain no repetitions.

Surely, for presenting a full cyclic near Hamiltonian cycle system of the $4K_v$, $CNHC(4K_v, C_{v-1})$, it is sufficient to provide a set of starter set that satisfies a near-4-factor. We give here example to explain the above definition.

Example 1. Let $G = 4K_{14}$ and $\mathcal{F} = \{C_{13}, C_{13}^*\}$ is a set of 13-cycles of G such that:

$$C_{13} = (1, 13, 2, 12, 3, 11, 4, 5, 10, 6, 9, 7, 8),$$
$$C_{13}^* = (13, 8, 12, 9, 11, 10, 4, 3, 5, 2, 6, 1, 7).$$

Firstly, it is easy to observe that each non zero element in Z_{14} occurs exactly twice in the 13-cycles of $\mathcal{F}$. Since, the cycle graph is 2-regular graph, then every vertex has a degree 4 except a zero element (isolated vertex) has a degree zero. Thus, it is satisfies the near-4-factor with focus zero element. Secondly, the list of differences set of the set $\mathcal{F}$ is listed in Table I

TABLE I
THE LIST OF DIFFERENCES OF $\mathcal{F}$

13-cycles	Difference set
(1,13,2,12,3,11,4,5,10,6,9,7,8)	{2,12,3,11,4,10,5,9,6,8,7,7,1,13, 5,9,4,10,3,11,2,12,1,13,7,7}
(13,8,12,9,11,10,4,3,5,2,6,1,7)	{5,9,4,10,3,11,2,12,1,13,6,8,1,13, 2,12,3,11,4,10,5,9,6,8,6,8}

It can be seen from the Table I, $\Delta(\mathcal{F}) = \Delta(C_{13}) \cup \Delta(C_{13}^*)$ covers each nonzero element in Z_{14} exactly four times. Since the cycles set $\mathcal{F}$ has trivial stabilizer based on Lemma 1, then the set $\mathcal{F} = \{C_{13}, C_{13}^*\}$ is the starter set of $CNHC(4K_{14}, C_{13})$ by Lemma 2.

Therefore, $CNHC(4K_{14}, C_{13})$ is an (14×2) array design and cycles set $\mathcal{F} = \{C_{13}, C_{13}^*\}$ in the first row generates all cycles in (14×2) array by repeated addition of 1 modular 14 as shown in the Table II.

TABLE II
$CNHC(4K_{14}, C_{13})$

Focus	$CNHC(4K_{14}, C_{13})$	
$r = 0$	(1,13,2,12,3,11,4,5,10,6,9,7,8)	(13,8,12,9,11,10,4,3,5,2,6,1,7)
$r = 1$	(2,0,3,13,4,12,5,6,11,7,10,8,9)	(0,9,13,10,12,11,5,4,6,3,7,2,8)
$r = 2$	(3,1,4,0,5,13,6,7,12,8,11,9,10)	(1,10,0,11,13,12,6,5,7,4,8,3,9)
⋮	⋮	⋮
$r = 13$	(0,12,1,1,2,10,3,4,9,5,8,6,7)	(12,7,11,8,10,9,3,2,4,1,5,0,6)

Throughout the paper, a near Hamiltonian cycle of order $(v-1)$ will be represented as connected paths, we mean that $C_{v-1} = \left(c_{(1)}, P_{(1,2)}^{2n}, P_{(3,4)}^{2n}\right)$ where, $P_{(1,2)}^{2n}$ and $P_{(3,4)}^{2n}$ are $(2n)$-paths such that:

$$\begin{aligned} P_{(1,2)}^{2n} &= \left[c_{(1,1)}, c_{(2,1)}, c_{(1,2)}, c_{(2,2)}, \dots, c_{(1,n)}, c_{(2,n)}\right], \\ &= \left[\cup_{i=1}^{n} c_{(1,i)}, c_{(2,i)}\right] \\ P_{(3,4)}^{2n} &= \left[c_{(3,1)}, c_{(4,1)}, c_{(3,2)}, c_{(4,2)}, \dots, c_{(3,n)}, c_{(4,n)}\right] \\ &= \left[\cup_{i=1}^{n} c_{(3,i)}, c_{(4,i)}\right]. \end{aligned}$$

Let the vertex sets of $P_{(1,2)}^{2n}$ and $P_{(3,4)}^{2n}$ are $\{\cup_{i=1}^{n} c_{(1,i)}, \cup_{i=1}^{n} c_{(2,i)}\}, \{\cup_{i=1}^{n} c_{(3,i)}, \cup_{i=1}^{n} c_{(4,i)}\}$, respectively. And the list of difference sets of $P_{(1,2)}^{2n}$ and $P_{(3,4)}^{2n}$ will be calculated as follows:

$$\Delta\left(P_{(1,2)}^{2n}\right) = \Delta_1\left(P_{(1,2)}^{2n}\right) \cup \Delta_2\left(P_{(1,2)}^{2n}\right),$$
$$\Delta\left(P_{(3,4)}^{2n}\right) = \Delta_1\left(P_{(3,4)}^{2n}\right) \cup \Delta_2\left(P_{(3,4)}^{2n}\right) \text{ such that}$$
$$\Delta_1\left(P_{(1,2)}^{2n}\right) = \left\{\pm\left|c_{(1,i)} - c_{(2,i)}\right| \middle| 1 \le i \le n\right\}.$$
$$\Delta_2\left(P_{(1,2)}^{2n}\right) = \left\{\pm\left|c_{(1,i+1)} - c_{(2,i)}\right| \middle| 1 \le i \le n-1\right\}.$$
$$\Delta_1\left(P_{(3,4)}^{2n}\right) = \left\{\pm\left|c_{(3,i)} - c_{(4,i)}\right| \middle| 1 \le i \le n\right\}.$$
$$\Delta_2\left(P_{(3,4)}^{2n}\right) = \left\{\pm\left|c_{(3,i+1)} - c_{(4,i)}\right| \middle| 1 \le i \le n-1\right\}.$$

And we define $\Delta\left(c_{(1)}, P^{2n}_{(1,2)}\right), \Delta\left(P^{2n}_{(3,4)}, c_{(1)}\right)$ and $\Delta\left(P^{2n}_{(1,2)}, P^{2n}_{(3,4)}\right)$ as follows

$\Delta\left(c_{(1)}, P^{2n}_{(1,2)}\right) = \pm\left|c_{(1)} - c_{(1,1)}\right|,$

$\Delta\left(P^{2n}_{(3,4)}, c_{(1)}\right) = \pm\left|c_{(4,n)} - c_{(1)}\right|$

$\Delta\left(P^{2n}_{(1,2)}, P^{2n}_{(3,4)}\right) = \pm\left|c_{(2,n)} - c_{(3,1)}\right|.$

So, the list of difference of C_{v-1} shall be represented as a follows:

$$\Delta(C_{v-1}) = \Delta\left(P^{2n}_{(1,2)}\right) \cup \Delta\left(P^{2n}_{(3,4)}\right) \cup \Delta\left(c_{(1)}, P^{2n}_{(1,2)}\right) \cup \Delta\left(P^{2n}_{(3,4)}, c_{(1)}\right) \cup \Delta\left(P^{2n}_{(1,2)}, P^{2n}_{(3,4)}\right)$$

.

Now we are able to provide our main result.

Theorem 1. There exists a full cyclic near Hamiltonian cycle system of $4K_v$, $CNHC(4K_v, C_{v-1})$, when $v = 4n+2, n > 2$.

Proof. Suppose $\mathcal{F} = \{C_{4n+1}, C^*_{4n+1}\}$ is a set of near Hamiltonian cycles of $4K_{4n+2}$ where

$C_{4n+1} = \left(1, P^{2n}_{(1,2)}, P^{2n}_{(3,4)}\right),$

$C^*_{4n+1} = \left(2n+1, P^{(2n)^*}_{(1,2)}, P^{(2n)^*}_{(3,4)}\right)$

Such that:

- $P^{2n}_{(1,2)} = [4n+1, 2, 4n, 3, \ldots, 3n+2, n+1].$
  $= [\cup_{i=1}^{n} 4n+2-i, i+1]$
- $P^{2n}_{(3,4)} = [n+2, 3n+1, n+3, 3n, \ldots, 2n+1, 2n+2] == [\cup_{i=1}^{n} n+1+i, 3n+2-i]$
- $P^{(2n)^*}_{(1,2)} = [4n+1, 2n+2, 4n, 2n+3, \ldots, 3n+2, 3n+1] = [\cup_{i=1}^{n} 4n+2-i, 2n+1+i]$
  .
- $P^{(2n)^*}_{(3,4)} = [n+1, n, n+2, n-1, \ldots, 2n, 1]$
  $= [\cup_{i=1}^{n} n+i, n+1-i].$

We will divide the proof into two parts as follows:

Part 1. In this part will be proved that $\mathcal{F}$ satisfies a near-4-factor. We shall calculate the vertex set of C_{4n+1} and C^*_{4n+1} such that:

$V(C_{4n+1}) = V\left(P^{2n}_{(1,2)}\right) \cup V\left(P^{2n}_{(3,4)}\right) \cup \{1\}.$

$V(C^*_{4n+1}) = V\left(P^{(2n)^*}_{(1,2)}\right) \cup V\left(P^{(2n)^*}_{(3,4)}\right) \cup \{2n+1\}.$

$$\cup_{i=1}^{n} c_{(1,i)} = \{4n+2-i, 1 \le i \le n\} = \{4n+1, 4n, \ldots, 3n+2\}, \quad (1)$$

$$\cup_{i=1}^{n} c_{(2,i)} = \{i+1, 1 \le i \le n\} = \{2, 3, \ldots, n+1\}, \quad (2)$$

$$\cup_{i=1}^{n} c_{(3,i)} = \{n+1+i, 1 \le i \le n\} = \{n+2, n+3, \ldots, 2n+1\}, \quad (3)$$

$$\cup_{i=1}^{n} c_{(4,i)} = \{3n+2-i, \quad 1 \le i \le n\} = \{3n+1, 3n, \ldots, 2n+2\}. \quad (4)$$

From above equations, it is easy to notice that $V(C_{4n+1})$ covers each nonzero element of Z_{4n+2} exactly once.

$$\cup_{i=1}^{n} c^*_{(1,i)} = \{4n+2-i, 1 \le i \le n\} = \{4n+1, 4n, \ldots, 3n+2\}, \quad (5)$$

$$\cup_{i=1}^{n} c^*_{(2,i)} = \{2n+1+i, 1 \le i \le n\} = \{2n+2, 2n+3, \ldots, 3n+1\}, \quad (6)$$

$$\cup_{i=1}^{n} c^*_{(3,i)} = \{n+i, 1 \le i \le n\} = \{n+1, n+2, \ldots, 2n\} \quad (7)$$

$$\cup_{i=1}^{n} c^*_{(4,i)} = \{n+1-i\} = \{n, n-1, \ldots, 1\} \quad (8)$$

It can be observed from the above equations that $V(C_{4n+1}) = V(C^*_{4n+1})$. Then, the multiset $V(C_{4n+1}) \cup V(C^*_{4n+1})$ covers each nonzero elements of Z_{4n+2} exactly twice. Since the cycle graph is 2-regular graph, therefore $\mathcal{F} = \{C_{4n+1}, C^*_{4n+1}\}$ satisfies a near-four-factor (with focus zero).

Part 2. In this part we will prove $\mathcal{F} = \{C_{4n+1}, C^*_{4n+1}\}$ is the starter set of cyclic $(v-1)$-cycle system of $4K_v$. So, we will calculate the difference set of each of them as follows:

$$\Delta(C_{4n+1}) = \Delta\left(P^{2n}_{(1,2)}\right) \cup \Delta\left(P^{2n}_{(3,4)}\right) \cup \Delta\left(c_{(1)}, P^{2n}_{(1,2)}\right) \cup \Delta\left(P^{2n}_{(1,2)}, P^{2n}_{(3,4)}\right) \cup \Delta\left(P^{2n}_{(3,4)}, c_{(1)}\right) \quad (9)$$

- $\Delta_1\left(P^{2n}_{(1,2)}\right) = \cup_{i=1}^{n} \pm|4n+1-2i| = \{4n-1, 4n-3, \ldots, 2n+1\} \cup \{3, 5, \ldots, 2n+1\}$
  .
- $\Delta_2\left(P^{2n}_{(1,2)}\right) = \cup_{i=1}^{n-1} \pm|4n-2i| = \{4n-2, 4n-4, \ldots, 2n+2\} \cup \{4, 6, \ldots, 2n\}$
  .
- $\Delta_1\left(P^{2n}_{(3,4)}\right) = \cup_{i=1}^{n} \pm|2n+1-2i| = \{2n-1, 2n-3, \ldots, 1\} \cup \{2n+3, 2n+5, \ldots, 4n+1\}$
  .
- $\Delta_2\left(P^{2n}_{(3,4)}\right) = \cup_{i=1}^{n-1} \pm|2n-2i| = \{2n-2, 2n-4, \ldots, 2\} \cup \{2n+4, 2n+6, \ldots, 4n\}$
- $\Delta\left(c_{(1)}, P^{2n}_{(1,2)}\right) = \pm\left|c_{(1)} - c_{(1,1)}\right| = \{2, 4n\},$
- $\Delta\left(P^{2n}_{(1,2)}, P^{2n}_{(3,4)}\right) = \pm\left|c_{(2,n)} - c_{(3,1)}\right| = \{1, 4n+1\}.$
- $\Delta\left(P^{2n}_{(3,4)}, c_{(1)}\right) = \pm\left|c_{(4,n)} - c_{(1)}\right| = \{2n+1, 2n+1\}$

From the equation 9, We note that the list of differences of C_{4n+1}, $\Delta(C_{4n+1})$, covers each nonzero elements of Z_{4n+2} twice except the differences $\{2n, 2n+2\}$ appear once.

Now we will calculate $\Delta(C^*_{4n+1})$ such as

$$\Delta(C^*_{4n+1}) = \Delta\left(P^{(2n)^*}_{(1,2)}\right) \cup \Delta\left(P^{(2n)^*}_{(3,4)}\right) \cup \Delta\left(c^*_{(1)}, P^{(2n)^*}_{(1,2)}\right) \cup \Delta\left(P^{(2n)^*}_{(1,2)}, P^{(2n)^*}_{(3,4)}\right) \cup \Delta\left(P^{(2n)^*}_{(3,4)}, c^*_{(1)}\right) \quad (10)$$

- $\Delta_1\left(P^{(2n)^*}_{(1,2)}\right) = \cup_{i=1}^{n} \pm|2n+1-2i| = \{2n-1, 2n-3, \ldots, 1\} \cup \{2n+3, 2n+5, \ldots, 4n+1\}$
  .

- $\Delta_2\left(P_{(1,2)}^{(2n)^*}\right) = \cup_{i=1}^{n-1} \pm|2n-2i| = \{2n-2, 2n-4, \dots, 2\} \cup \{2n+4, 2n+6, \dots, 4n\}$.
- $\Delta_1\left(P_{(3,4)}^{(2n)^*}\right) = \cup_{i=1}^{n} \pm|2i-1| = \{1, 3, \dots, 2n-1\} \cup \{4n+1, 4n-1, \dots, 2n+3\}$.
- $\Delta_2\left(P_{(3,4)}^{(2n)^*}\right) = \cup_{i=1}^{n-1} \pm|2i| = \{2, 4, \dots, 2n-2\} \cup \{4n, 4n-2, \dots, 2n+4\}$.
- $\Delta\left(c_{(1)}^*, P_{(1,2)}^{(2n)^*}\right) = \pm\left|c_{(1)}^* - c_{(1,1)}^*\right| = \{2n, 2n+2\}$.
- $\Delta\left(P_{(1,2)}^{(2n)^*}, P_{(3,4)}^{(2n)^*}\right) = \pm\left|c_{(2,n)}^* - c_{(3,1)}^*\right| = \{2n, 2n+2\}$.
- $\Delta\left(P_{(3,4)}^{(2n)^*}, c_{(1)}^*\right) = \pm\left|c_{(4,n)}^* - c_{(1)}^*\right| = \{2n, 2n+2\}$.

As clearly shown, in the equations 10, every nonzero element in Z_{4n+2} appears twice except $\{2n, 2n+2\}$ appear three times in $\Delta(C_{4n+1}^*)$. Based on Lemma 1, the cycles $\{C_{4n+1}, C_{4n+1}^*\}$ have trivial stabilizer.

One can easily note that $\Delta(\mathcal{F}) = \Delta(C_{4n+1}) \cup \Delta(C_{4n+1}^*)$ covers each non zero integers in Z_{4n+2} four times. Thus, $\mathcal{F} = \{C_{4n+1}, C_{4n+1}^*\}$ is the starter cycles of cyclic $(v-1)$-cycle system of $4K_v$ by Lemma 2. Hence, the cycles set $\mathcal{F} = \{C_{4n+1}, C_{4n+1}^*\}$ generates a full near Hamiltonian cycle system of $4K_v$ by adding one modular v when $v = 4n+2,\ n > 2$

IV. CYCLIC HAMILTONIAN WHEEL SYSTEM

A wheel graph of order m, denoted by W_m, consists of a singleton graph K_1 and a cycle graph of order $m-1$, C_{m-1}, in which the K_1 is connected to all the vertices of C_{m-1}, written $K_1 + C_{m-1}$ or $c_0 + (c_1, c_2, \dots, c_{m-1})$. An m-wheel contains $2(m-1)$ edges such that the edge set of W_m is $E(W_m) = E(K_{1,(m-1)}) \cup E(C_{m-1})$ [12].

An m-wheel system of graph G is a decomposition of edge set of G into collection $\mathcal{W} = \{W_{m_1}, \dots, W_{m_r}\}$ of edges-disjoint of m-wheels. Similar to the cyclic cycle system, an m-wheel system of λK_v is a cyclic if $V(\lambda K_v) = Z_v$ and if $W_m = c_0 + (c_1, c_2, \dots, c_{m-1}) \in \mathcal{W}$ implies that $W_m + 1 = (c_0+1) + (c_1+1, c_2+1, \dots, c_{m-1}+1)$ is also in $\mathcal{W}$. Moreover, if $m = v$ then it is called a cyclic Hamiltonian wheel system. The list of difference of $W_m = c_0 + (c_1, c_2, \dots, c_{m-1})$ is $\Delta(W_m) = \Delta(K_{1,(m-1)}) \cup \Delta(C_{m-1})$ such that $\Delta(C_m) = \{\pm|c_{i+1} - c_i| \mid 1 \le i \le m-1\}$ where $c_m = c_0$ and $\Delta\left(K_{(1,(m-1))}\right) = \{\pm|c_i - c_0| \mid 1 \le i \le m-1\}$. More generally, given a multiset $\mathcal{W} = \{W_{m_1}, \dots, W_{m_t}\}$ of m-wheels of λK_v, the list of differences from $\mathcal{W}$ is $\Delta\mathcal{W} = \cup_{i=1}^{t} \Delta W_{m_i}$.

Definition 2. The full cyclic Hamiltonian wheel system of a graph $8K_v$, denoted by $CHWS(8K_v, \mathcal{W})$, is a cyclic (v)-wheel system of a graph $8K_v$ that generated by starter set $\mathcal{W} = \{W_{v_1}, \dots, W_{v_r}\}$ such that the associated cycles with wheels satisfy near-four-factor with focus singleton graph.

In other words $CHWS(8K_v, \mathcal{W})$ is a $(v \times |\mathcal{W}|)$ array such that satisfies the following conditions:

1) The wheels in row r form a r +(near-four-factor).
2) The wheels associated with the rows contain no repetitions.

For clarity, we provide an example to demonstrate the construction of a cyclic Hamiltonian wheel system stated above.

Example 2. Let $G = 8K_{14}$ and $\mathcal{W} = \{W_{14}, W_{14}^*\}$ is a set of Hamiltonian wheels of $8K_{14}$. Where

$W_{14} = K_1 + C_{13} = 0 + (1, 13, 2, 12, 3, 11, 4, 5, 10, 6, 9, 7, 8)$

and

$W_{14}^* = K_1^* + C_{13}^* = 0 + (13, 8, 12, 9, 11, 10, 4, 3, 5, 2, 6, 1, 7)$

From the Example 1, we can note that the 13-cycles $\{C_{13}, C_{13}^*\}$ satisfy a near-four-factor with focus 0 (zero element). Moreover, the list of differences from $\{C_{13}, C_{13}^*\}$ covers every nonzero element of Z_{14} exactly four times. Now we want to find the list of differences from $(K_{1,(13)}) \cup (K_{1,(13)}^*)$ as a follows

$$\Delta(K_{1,(13)}) = \{\pm|c_i - c_0| \mid 1 \le i \le 13\} = \{\pm|c_i| \mid 1 \le i \le 13\}$$

such that the $c_i \in V(C_{13}), 1 \le i \le 13$. Since the vertex set of C_{13} is $Z_{14} - \{0\}$ and $Z_{14} = -Z_{14}$, then $\Delta(K_{1,(13)})$ covers each nonzero element of Z_{14} twice. Similarly, one can be found $\Delta(K_{1,(13)}^*) = \Delta(K_{1,(13)})$. So $\Delta(\mathcal{W})$ covers each nonzero of Z_{14} four times. Thus, $\mathcal{W} = \{W_{14}, W_{14}^*\}$ is the starter set of $CHWS(8K_{14}, \mathcal{W})$.

Then $CHWS(8K_{14}, W_{14})$ is an (14×2) array deign where all its wheels can be generated by repeated addition 1 (modular 14) on the starter set $\mathcal{W}$ as shown in the Table III.

TABLE III
$CHWS(8K_{14}, W_{14})$

$CHWS(8K_{14}, \mathcal{W})$	
$0 + (1,13,2,12,3,11,4,5,10,6,9,7,8)$	$0 + (13,8,12,9,11,10,4,3,5,2,6,1,7)$
$1 + (2,0,3,13,4,12,5,6,11,7,10,8,9)$	$1 + (0,9,13,10,12,11,5,4,6,3,7,2,8)$
$2 + (3,1,4,0,5,13,6,7,12,8,11,9,10)$	$2 + (1,10,0,11,13,12,6,5,7,4,8,3,9)$
⋮	⋮
$13 + (0,12,1,1,2,10,3,4,9,5,8,6,7)$	$13 + (12,7,11,8,10,9,3,2,4,1,5,0,6)$

The following theorem proves the existence of $CHWS(8K_{4n+2}, \mathcal{W})$.

Theorem 2. There exists a full cyclic Hamiltonian wheel system of $8K_v$, $CHWS(8K_v, \mathcal{W})$, for $v = 4n+2, n > 2$.

Proof. We have to present a starter set $\mathcal{W} = \{K_1 + C_{4n+1}, K_1^* + C_{4n+1}^*\}$ of $CHWS(8K_v, \mathcal{W})$ such that the cycles associated with the wheels in $\mathcal{W}$ satisfy a near-four-factor with focus a singleton graph.

Suppose $\mathcal{W} = \{0 + C_{4n+1}, 0 + C^*_{4n+1}\}$ is a set of Hamiltonian wheels of $8K_{4n+2}$ where

$$C_{4n+1} = \left(1, P^{2n}_{(1,2)}, P^{2n}_{(3,4)}\right),$$
$$C^*_{4n+1} = \left(2n + 1, P^{(2n)^*}_{(1,2)}, P^{(2n)^*}_{(3,4)}\right)$$

Such that:

- $P^{2n}_{(1,2)} = [4n + 1, 2, 4n, 3, \ldots, 3n + 2, n + 1]$.
- $P^{2n}_{(3,4)} = [n + 2, 3n + 1, n + 3, 3n, \ldots, 2n + 1, 2n + 2]$.
- $P^{(2n)^*}_{(1,2)} = [4n + 1, 2n + 2, 4n, 2n + 3, \ldots, 3n + 2, 3n + 1]$.
- $P^{(2n)^*}_{(3,4)} = [n + 1, n, n + 2, n - 1, \ldots, 2n, 1]$.

From Theorem 1, the cycles associated with the Hamiltonian wheels in $\mathcal{W}$ satisfy the near-four-factor with focus zero element.

Now, we want to prove $\mathcal{W} = \{K_1 + C_{4n+1}, K^*_1 + C^*_{4n+1}\}$ is a $(\lambda K_v, \mathcal{W})$-difference system. To do this, it is enough to show that the list of differences

$$\Delta\mathcal{W} = \{\Delta(C_{4n+1}) \cup \Delta(C^*_{4n+1}) \cup \Delta\left(K^*_{1,(4n+1)}\right) \cup \Delta\left(K_{1,(4n+1)}\right)\}$$

covers each element of $\{Z_{4n+2} - \{0\}\}$ eight times. Firstly, as indicated in Theorem 1, the list of differences of $\{(C_{4n+1}) \cup (C^*_{4n+1})\}$ cover each nonzero element in Z_{4n+2} exactly four times.

Secondly, the list of differences of $\left(K_{1,(4n+1)}\right)$ is $\{\pm|c_i - 0| \mid c_i \in C_{4n+1}\}$. Since $V(C_{4n+1}) = Z_{4n+2} - \{0\}$ then $\{|c_i - 0| \mid c_i \in C_{4n+1}\} = Z_{4n+2} - \{0\}$. Because of $Z_{4n+2} = -\{Z_{4n+2}\}$, then $\Delta\left(K_{1,(4n+1)}\right) = \{\pm|c_i - 0| \mid c_i \in C_{4n+1}\}$ covers each nonzero element of Z_{4n+2} twice. Likewise, we repeat the same strategy on cycle $K^*_{1,(4n+1)}$ to find $\Delta\left(K^*_{1,(4n+1)}\right)$. Also, it is an easy matter to check that $\Delta\left(K^*_{1,(4n+1)}\right) = \Delta\left(K_{1,(4n+1)}\right)$.

Linking together the above list of differences, we see that $\Delta\mathcal{W}$ covers each nonzero element of Z_{4n+2} eight times. On the other hand, each wheel graph in $\mathcal{W}$ has trivial stabilizer based on Lemma 1. Therefore, $\mathcal{W}$ is the starter set of $CHWS(8K_v, \mathcal{W})$, by Lemma 2. One can be generated $CHWS(8K_v, \mathcal{W})$ by repeated addition 1 modular v on $\mathcal{W}$. □

V. Conclusion

In this paper, we have provided new designs $CNHC(4K_v, C_{v-1})$ and $CHWS(8K_v, W_v)$ where $v \equiv 2 (mod 4)$. These designs are interested in a decomposition of complete multigraph into cyclic $(v-1)$-cycle and cyclic (v)-wheel graphs, respectively. We have also proved the existence of these designs by constructed the starter set for each of them. Moreover, one can ask if $CNHC(2\lambda K_v, C_{v-1})$ and $CHWS(2\lambda K_v, W_v)$ can be constructed for the case $v \equiv 2, 4 (mod\ 4)$ and $\lambda > 2$

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