Decompositions of Complete Multigraphs into Cyclic Designs

Mowafaq Alqadri#1, Haslinda Ibrahim#2, Sharmila Karim#3

# Department of Mathematics, School of Quantitive Sciences, College of Arts and Sciences, Universiti Utara Malaysia, Sintok, Kedah, 06010, Malaysia  
 E-mail: 1moufaqq@yahoo.com; 2linda@uum.edu.my; 3mila@uum.edu.my

**Abstract**— Let and be positive integer, denote a complete multigraph. A decomposition of a graph is a set of subgraphs of whose edge sets partition the edge set of . The aim of this paper, is to decompose a complete multigraph into cyclic -cycle system according to specified conditions. As the main consequence, construction of decomposition of into cyclic Hamiltonian wheel system, where , is also given. The difference set method is used to construct the desired designs

**Keywords**— Cyclic design; Hamiltonian cycle; Near four factor; Wheel graph.

# Introduction

Throughout this paper, all graphs consider finite and undirected. A complete graph of order denotes by . An -design is a decomposition of the graph into subgraphs belonging to an assigned multiset .

An -cycle, written, consists of distinct vertices and edges and and -cycle of a graph is called Hamiltonian when its vertices passes through all the vertex set of . An -path, written , consists of distinct vertices and edges . An -cycle system of a graph is -design where is a collection of -cycles. If then such -cycle system is called -cycle system of order and is also said a simple when its cycles are all distinct.

An automorphism group on -design is a bijections on fixed . An -design is a cyclic if it admit automorphism group acting regularly on [1]. For a cyclic -design, we can assume that . So, the automorphism can be represented by

or

A starter set of a cyclic -design is a set of subgraphsof that generates all subgraphs of by repeated addition of modular .

A complete multigraph of order , denoted by , is obtained by replacing each edge of with edges. The problem which concerned in the decomposition of the complete multigraph into subgraphs has received much attention in recent years. The necessary and sufficient conditions for decomposing into cycles of order and cycles of prime oredr have been established by [2]. While, the existence theorem of -cycle system of has been proved for all values of in [3]. For the important case of , the existence question for -cycle system of order has been completely settled by [4] in the case odd and by [ 5] in the case even. Moreover, the cyclic -cycle system of order for , denoted by , has been constructed by [6] and for a cyclic Hamiltonian cycle system of order was proved when is an odd integer but and with a prime and [7].

On the other hand, the necessary and sufficient conditions for decomposing into cycle and star graphs have been investigated by [8].

A four-factor of a graph is a spanning subgraph whose vertices have a degree . While a near-four-factor is a spanning subgraph in which all vertices have a degree four with exception of one vertex (isolated vertex) which has a degree zero [9].

In this paper, we propose a new type of cyclic cycle system that is called cyclic near Hamiltonian cycle system of , denoted . This is obtained by combination a near-four-factors and cyclic -cycle system of when . Furthermore, the construction of will be employed to decompose into Hamiltonian wheels.

# Preliminaries

In our paper, all graphs considered have vertices in . We will use the difference set method to construct the desired designs. The difference between any two distinct vertices and in is , arithmetic . Given an -cycle, the differences from are the multiset where . Let be an -cycles of the list of differences from is .

The orbit of cycle , denoted by , is the set of all distinct -cycles in the collection . The length of is its cardinality, i.e., where is the minimum positive integer such that . A cycle orbit of length on is said full and otherwise short. [10]

The stabilizer of a subgraph of a graph of order is and has trivial stabilizer when. One may easily deduce the following result.

For presenting a cyclic -cycle system of, it sufficient to construct a starter set, i.e., -cycle system of representations for its cycle orbits. As particular consequences of the theory developed in [11] we have:

Lemma 1. Let be a subgraph of and . Then each nonzero integer in appears a multiple of times.

Lemma 2. Let be a multiset of subgraphs of and every subgraph of has trivial stabilizer. Then is a starter of cyclic-design if and only if covers each nonzero integer of exactly times.

# Cyclic Near Hamiltonian cycle System

Definition 1. A full cyclic near Hamiltonian cycle system of the , denoted by , is a cyclic -cycle system of graph, that satisfies the following conditions:

1. The cycle in row form a near-4-factor with focus .
2. The cycles associated with the rows contain no repetitions.

Surely, for presenting a full cyclic near Hamiltonian cycle system of the , , it is sufficient to provide a set of starter set that satisfies a near-4-factor. We give here example to explain the above definition.

Example 1. Let and is a set of -cycles of such that:

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Firstly, it is easy to observe that each non zero element in occurs exactly twice in the -cycles of . Since, the cycle graph is -regular graph, then every vertex has a degree except a zero element (isolated vertex) has a degree zero. Thus, it is satisfies the near-4-factor with focus zero element. Secondly, the list of differences set of the set is listed in Table I.

TABLE I

THE LIST OF DIFFERENCES OF

|  |  |
| --- | --- |
| **-cycles** | **Difference set** |
|  |  |
|  |  |

It can be seen from the Table I, covers each nonzero element in exactly four times. Since the cycles set has trivial stabilizer based on Lemma 1, then the set is the starter set of by Lemma 2.

Therefore, is an array design and cycles set in the first row generates all cycles in array by repeated addition of modular as shown in the Table II.

TABLE II

|  |  |  |
| --- | --- | --- |
| **Focus** |  | |
|  |  |  |
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|  |  |  |
|  |  |  |
|  |  |  |

Throughout the paper, a near Hamiltonian cycle of order will be represented as connected paths, we mean that where and are -paths such that:

*,*

*.*

*.*

Let the vertex sets of and are , respectively. And the list of difference sets of and will be calculated as follows:

*,*

such that

*.*

*.*

*.*

.

And we define and as follows

*,*

.

So, the list of difference of shall be represented as a follows:

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Now we are able to provide our main result.

Theorem 1.There exists a full cyclic near Hamiltonian cycle system of , , when .

Proof. Suppose is a set of near Hamiltonian cycles of where

*,*

Such that:

* .

* .

.

We will divide the proof into two parts as follows:

Part 1. In this part will be proved that satisfies a near-4-factor. We shall calculate the vertex set of and such that:

*.*

*.*

, (1)

, (2)

, (3)

. (4)

From above equations, it is easy to notice that covers each nonzero element of exactly once.

, (5)

, (6)

(7)

(8)

It can be observed from the above equations that . Then, the multiset covers each nonzero elements of exactly twice. Since the cycle graph is -regular graph, therefore satisfies a near-four-factor (with focus zero).

Part 2. In this part we will prove is the starter set of cyclic -cycle system of . So, we will calculate the difference set of each of them as follows:

(9)

* + .
  + .
  + .
  + ,
  + .

From the equation 9, We note that the list of differences of *,* , covers each nonzero elements of twice except the differences appear once.

Now we will calculate such as

(10)

* .
* .
* .
* .
* .
* .
* .

As clearly shown, in the equations 10, every nonzero element in appears twice except appear three times in . Based on Lemma 1, the cycles have trivial stabilizer.

One can easily note that covers each non zero integers in four times. Thus, is the starter cycles of cyclic -cycle system of by Lemma 2. Hence, the cycles set generates a full near Hamiltonian cycle system of by adding one modular when □

# cyclic Hamiltonian wheel system

A wheel graph of order , denoted by , consists of a singleton graph and a cycle graph of order , , in which the is connected to all the vertices of , written or . An -wheel contains edges such that the edge set of is [12].

An -wheel system of graph is a decomposition of edge set of into collection of edges- disjoint of -wheels. Similar to the cyclic cycle system, an -wheel system of is a cyclic if and if implies that is also in . Moreover, if then it is called a cyclic Hamiltonian wheel system. The list of difference of is such that where and . More generally, given a multiset of -wheels of , the list of differences from is .

Definition 2. The full cyclic Hamiltonian wheel system of a graph, denoted by, is a cyclic -wheel system of a graph that generated by starter set such that the associated cycles with wheels satisfy near-four-factor with focus singleton graph.

In other words is a array such that satisfies the following conditions:

1. The wheels in row form a (near-four-factor).
2. The wheels associated with the rows contain no repetitions.

For clarity, we provide an example to demonstrate the construction of a cyclic Hamiltonian wheel system stated above.

Example 2. Let and is a set of Hamiltonian wheels of . Where and . From the Example , we can note that the -cycles satisfy a near-four-factor with focus 0 (zero element). Moreover, the list of differences from covers every nonzero element of exactly four times. Now we want to find the list of differences from as a follows

such that the . Since the vertex set of is and , then covers each nonzero element of twice. Similarly, one can be found . So covers each nonzero of four times. Thus, is the starter set of .

Then is an array deign where all its wheels can be generated by repeated addition 1 (modular ) on the starter set as shown in the Table III.

TABLE III

|  |  |
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|  |  |
|  |  |

The following theorem proves the existence of .

Theorem 2. There exists a full cyclic Hamiltonian wheel system of , , for .

Proof. We have to present a starter set of such that the cycles associated with the wheels in satisfy a near-four-factor with focus a singleton graph.

Suppose is a set of Hamiltonian wheels of where

,

Such that:

* .
* .
* .
* .

From Theorem 1, the cycles associated with the Hamiltonian wheels in satisfy the near-four-factor with focus zero element.

Now, we want to prove is a -difference system. To do this, it is enough to show that the list of differences

covers each element of eight times. Firstly, as indicated in Theorem 1, the list of differences of cover each nonzero element in exactly four times.

Secondly, the list of differences of is . Since then . Because of , then covers each nonzero element of twice. Likewise, we repeat the same strategy on cycle to find . Also, it is an easy matter to check that .

Linking together the above list of differences, we see that covers each nonzero element of eight times. On the other hand, each wheel graph in has trivial stabilizer based on Lemma 1. Therefore, is the starter set of , by Lemma 2. One can be generated by repeated addition 1 modular on . □

# Conclusion

In this paper, we have provided new designs and where . These designs are interested in a decomposition of complete multigraph into cyclic -cycle and cyclic -wheel graphs, respectively. We have also proved the existence of these designs by constructed the starter set for each of them. Moreover, one can ask if and can be constructed for the case and .

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