Decompositions of Complete Multigraphs into Cyclic Designs

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**Abstract**— Let $v$ and $λ$ be positive integer, $λK\_{v}$ denote a complete multigraph. A decomposition of a graph $G$ is a set of subgraphs of $G$ whose edge sets partition the edge set of $G$. The aim of this paper, is to decompose a complete multigraph $4K\_{v}$ into cyclic $\left(v-1\right)$-cycle system according to specified conditions. As the main consequence, construction of decomposition of $8K\_{v}$ into cyclic Hamiltonian wheel system, where $v≡2\left(mod 4\right)$, is also given. The difference set method is used to construct the desired designs

**Keywords**— Cyclic design; Hamiltonian cycle; Near four factor; Wheel graph.

# Introduction

Throughout this paper, all graphs consider finite and undirected. A complete graph of order $v$ denotes by $K\_{v}$. An $\left(G, Y\right)$-design is a decomposition of the graph $G$ into subgraphs belonging to an assigned multiset $Y$.

 An $m$-cycle, written$ C\_{m}=\left(c\_{0}, …, c\_{m-1}\right)$, consists of $m$ distinct vertices $\left\{c\_{0}, c\_{1}, …, c\_{m-1}\right\}$ and $m$ edges $\left\{c\_{i}c\_{i+1}\right\}, 0\leq i \leq m-2$ and $c\_{0}c\_{m-1}$ and $m$-cycle of a graph $G$ is called Hamiltonian when its vertices passes through all the vertex set of $G$. An $m$-path, written $\left[c\_{0}, …, c\_{m-1}\right]$, consists of $m$ distinct vertices $\left\{c\_{0}, c\_{1}, …, c\_{m-1}\right\}$ and $m-1$ edges $\left\{c\_{i}c\_{i+1}\right\}, 0\leq i \leq m-2$. An $m$-cycle system of a graph $G$ is $\left(G, C\right)$-design where $C$ is a collection of $m$-cycles. If $G=K\_{v}$ then such $m$-cycle system is called $m$-cycle system of order $v$ and is also said a simple when its cycles are all distinct.

 An automorphism group on $\left(G, Y\right)$-design is a bijections on $V\left(G\right)$ fixed $Y$. An $\left(G, Y\right)$-design is a cyclic if it admit automorphism group acting regularly on $V\left(G\right)$ [1]. For a cyclic $\left(G, Y\right)$-design, we can assume that $V\left(G\right)=Z\_{v}$. So, the automorphism can be represented by

$α:i\rightarrow i+1 \left(mod v\right)$ or $α:\left(0, 1, …, v-1\right)$

A starter set of a cyclic $\left(G, Y\right)$-design is a set of subgraphs$ $of $G$ that generates all subgraphs of $Y$ by repeated addition of $1$ modular $v$.

A complete multigraph of order $v$, denoted by $λK\_{v}$, is obtained by replacing each edge of $K\_{v}$ with $λ$ edges. The problem which concerned in the decomposition of the complete multigraph into subgraphs has received much attention in recent years. The necessary and sufficient conditions for decomposing $λK\_{v}$ into cycles of order $λ$ and cycles of prime oredr have been established by [2]. While, the existence theorem of $m$-cycle system of $λK\_{v}$ has been proved for all values of $λ$ in [3]. For the important case of $λ=1$, the existence question for $m$-cycle system of order $v$ has been completely settled by [4] in the case $m$ odd and by [ 5] in the case $m$ even. Moreover, the cyclic $m$-cycle system of order $v$ for $m=3$, denoted by $CTS(v,λ)$, has been constructed by [6] and for a cyclic Hamiltonian cycle system of order $v$ was proved when $v$ is an odd integer but $v\ne 15$ and $v\ne p^{α}$ with $p$ a prime and $α>1$ [7].

On the other hand, the necessary and sufficient conditions for decomposing $λK\_{v}$ into cycle and star graphs have been investigated by [8].

A four-factor of a graph $G $is a spanning subgraph whose vertices have a degree $4$. While a near-four-factor is a spanning subgraph in which all vertices have a degree four with exception of one vertex (isolated vertex) which has a degree zero [9].

In this paper, we propose a new type of cyclic cycle system that is called cyclic near Hamiltonian cycle system of $4K\_{v}$, denoted $CNHC\left(4K\_{v},C\_{v-1}\right)$. This is obtained by combination a near-four-factors and cyclic $\left(v-1\right)$-cycle system of $4K\_{v}$ when $v≡2\left(mod 4\right)$. Furthermore, the construction of $CNHC\left(4K\_{v},C\_{v-1}\right)$ will be employed to decompose $8K\_{v}$ into Hamiltonian wheels.

# Preliminaries

In our paper, all graphs considered have vertices in $Z\_{v}$. We will use the difference set method to construct the desired designs. The difference between any two distinct vertices $a$ and $b$ in $λK\_{v}$ is $\pm \left|a-b\right|$, arithmetic $\left(mod v\right)$. Given $C\_{m}=\left(c\_{0}, …, c\_{m-1}\right)$ an $m$-cycle, the differences from $C\_{m} $are the multiset $∆\left(C\_{m}\right)=\left\{\pm \left|c\_{i}-c\_{i-1}\right|\right. \left|\left.i=1, 2, …, m\right\}\right.$ where $c\_{0}=c\_{m}$. Let $F=\left\{B\_{1}, B\_{2}, …, B\_{r}\right\}$ be an $m$-cycles of $λK\_{v}$ the list of differences from $F$ is $∆\left(F\right)=\bigcup\_{i=1}^{r}∆\left(B\_{i}\right)$.

 The orbit of cycle $C\_{m}$, denoted by $orb\left(C\_{m}\right)$, is the set of all distinct $m$-cycles in the collection $\left\{ i\in Z\_{v}\right\}$. The length of $orb\left(C\_{m}\right)$ is its cardinality, i.e., $orb\left(C\_{m}\right)=k$ where $k$ is the minimum positive integer such that $C\_{m}+k=C\_{m}$. A cycle orbit of length $v$ on $λK\_{v}$ is said full and otherwise short. [10]

The stabilizer of a subgraph $H$ of a graph $G$ of order $v$ is $stab\left(H\right)=\left\{z\in Z\_{v} \left| z+H=H\right.\right\}$ and $H$ has trivial stabilizer when$ stab\left(H\right)=\left\{0\right\}$. One may easily deduce the following result.

For presenting a cyclic $m$-cycle system of$ λK\_{v}$, it sufficient to construct a starter set, i.e., $m$-cycle system of representations for its cycle orbits. As particular consequences of the theory developed in [11] we have:

Lemma 1. Let $H$ be a subgraph of $G$ and $\left|stab\left(H\right)\right|>1$. Then each nonzero integer in $∆H$ appears a multiple of $\left|stab\left(H\right)\right|$ times.

Lemma 2. Let $δ$ be a multiset of subgraphs of $λK\_{v}$ and every subgraph of $δ$ has trivial stabilizer. Then $δ$ is a starter of cyclic$\left( λK\_{v}, Y\right)$-design if and only if $Δδ$ covers each nonzero integer of $Z\_{v}$ exactly $λ$ times.

# Cyclic Near Hamiltonian cycle System

Definition 1. A full cyclic near Hamiltonian cycle system of the $4K\_{v}$, denoted by $CNHC\left(4K\_{v},C\_{v-1}\right)$, is a cyclic $\left(v-1\right)$-cycle system of $4K\_{v}$ graph, that satisfies the following conditions:

1. The cycle in row $r$ form a near-4-factor with focus $r$.
2. The cycles associated with the rows contain no repetitions.

Surely, for presenting a full cyclic near Hamiltonian cycle system of the $4K\_{v}$, $CNHC\left(4K\_{v},C\_{v-1}\right)$, it is sufficient to provide a set of starter set that satisfies a near-4-factor. We give here example to explain the above definition.

Example 1. Let $G=4K\_{14}$ and $F=\left\{C\_{13}, C\_{13}^{\*}\right\}$ is a set of $13$-cycles of $G$ such that:

$$C\_{13}=\left(1, 13, 2, 12, 3, 11, 4, 5, 10, 6, 9, 7, 8\right),$$

$C\_{13}^{\*}=\left(13, 8, 12, 9, 11, 10, 4, 3, 5, 2, 6, 1, 7\right)$.

Firstly, it is easy to observe that each non zero element in $Z\_{14}$ occurs exactly twice in the $13$-cycles of $F$. Since, the cycle graph is $2$-regular graph, then every vertex has a degree $4$ except a zero element (isolated vertex) has a degree zero. Thus, it is satisfies the near-4-factor with focus zero element. Secondly, the list of differences set of the set $F$ is listed in Table I.

TABLE I

THE LIST OF DIFFERENCES OF $F$

|  |  |
| --- | --- |
| $13$**-cycles** | **Difference set** |
| $$\left(1,13,2,12,3,11,4,5,10,6,9,7,8\right)$$ | $$\left\{\begin{array}{c}2,12,3,11,4,10,5,9,6,8,7,7,1,13,\\5,9,4,10,3,11,2,12,1,13,7,7\end{array}\right\}$$ |
| $$\left(13,8,12,9,11,10,4,3,5,2,6,1,7\right)$$ | $$\left\{\begin{array}{c}5,9,4,10,3,11,2,12,1,13,6,8,1,13,\\2,12,3,11,4,10,5,9,6,8,6,8\end{array}\right\}$$ |

It can be seen from the Table I, $∆(F)=∆\left(C\_{13}\right)∪∆\left(C\_{13}^{\*}\right)$ covers each nonzero element in $Z\_{14}$ exactly four times. Since the cycles set $F$ has trivial stabilizer based on Lemma 1, then the set $F=\left\{C\_{13}, C\_{13}^{\*}\right\}$ is the starter set of $CNHC\left(4K\_{14}, C\_{13}\right)$ by Lemma 2.

Therefore, $CNHC\left(4K\_{14}, C\_{13}\right)$ is an $\left(14×2\right)$ array design and cycles set $F=\left\{C\_{13}, C\_{13}^{\*}\right\}$ in the first row generates all cycles in $\left(14×2\right)$ array by repeated addition of $1$ modular $14$ as shown in the Table II.

TABLE II

$$CNHC\left(4K\_{14}, C\_{13}\right)$$

|  |  |
| --- | --- |
| **Focus** | $$CNHC\left(4K\_{14}, C\_{13}\right)$$ |
| $$r=0$$ | $$\left(1,13,2,12,3,11,4,5,10,6,9,7,8\right)$$ | $$\left(13,8,12,9,11,10,4,3,5,2,6,1,7\right)$$ |
| $$r=1$$ | $$\left(2,0,3,13,4,12,5,6,11,7,10,8,9\right)$$ | $$\left(0,9,13,10,12,11,5,4,6,3,7,2,8\right)$$ |
| $$r=2$$ | $$\left(3,1,4,0,5,13,6,7,12,8,11,9,10\right)$$ | $$\left(1,10,0,11,13,12,6,5,7,4,8,3,9\right)$$ |
| $$\vdots $$ | $$\vdots $$ | $$\vdots $$ |
| $$r=13$$ | $$\left(0,12,1,1,2,10,3,4,9,5,8,6,7\right)$$ | $$\left(12,7,11,8,10,9,3,2,4,1,5,0,6\right)$$ |

Throughout the paper, a near Hamiltonian cycle of order $(v-1)$ will be represented as connected paths, we mean that $C\_{v-1}=\left(c\_{(1)}, P\_{(1,2)}^{2n}, P\_{(3,4)}^{2n}\right)$ where$, P\_{(1,2)}^{2n}$ and $P\_{(3,4)}^{2n}$ are $(2n)$-paths such that:

 $P\_{(1,2)}^{2n}=\left[c\_{\left(1, 1\right)}, c\_{(2, 1)}, c\_{(1, 2)}, c\_{\left(2, 2\right)}, …, c\_{(1, n)}, c\_{(2, n)}\right]$*,*

$=\left[\bigcup\_{i=1}^{n}c\_{\left(1, i\right)}, c\_{(2, i)}\right]$*.* $P\_{(3, 4)}^{2n}=\left[c\_{\left(3, 1\right)}, c\_{(4, 1)}, c\_{(3, 2)}, c\_{\left(4, 2\right)}, …, c\_{(3, n)}, c\_{(4, n)}\right]$

 $=\left[\bigcup\_{i=1}^{n}c\_{\left(3, i\right)}, c\_{(4, i)}\right]$*.*

Let the vertex sets of $P\_{(1,2)}^{2n}$ and $P\_{(3, 4)}^{2n}$ are $\left\{\bigcup\_{i=1}^{n}c\_{(1, i)}, \bigcup\_{i=1}^{n}c\_{(2, i)}\right\}, \left\{\bigcup\_{i=1}^{n}c\_{(3, i)} , \bigcup\_{i=1}^{n}c\_{(4, i)} \right\}$, respectively. And the list of difference sets of $ P\_{(1,2)}^{2n}$ and $P\_{(3,4)}^{2n}$ will be calculated as follows:

$∆\left( P\_{(1,2)}^{2n}\right)=∆\_{1}\left(P\_{(1,2)}^{2n}\right)∪∆\_{2}\left(P\_{(1,2)}^{2n}\right)$*,*

$∆\left( P\_{(3, 4)}^{2n}\right)=∆\_{1}\left(P\_{(3, 4)}^{2n}\right)∪∆\_{2}\left(P\_{(3, 4)}^{2n}\right)$such that

$∆\_{1}\left(P\_{(1,2)}^{2n}\right)=\left\{1\leq i\leq n\right\}$*.*

$∆\_{2}\left(P\_{(1,2)}^{2n}\right)=\left\{1\leq i\leq n-1\right\}$*.*

$∆\_{1}\left(P\_{(3, 4)}^{2n}\right)=\left\{1\leq i\leq n\right\}$*.*

$∆\_{2}\left(P\_{(3, 4)}^{2n}\right)=\left\{1\leq i\leq n-1\right\}$.

And we define $∆\left(c\_{(1)}, P\_{(1,2)}^{2n}\right), ∆\left( P\_{(3,4)}^{2n}, c\_{(1)}\right) $and $∆\left( P\_{(1,2)}^{2n}, P\_{(3, 4)}^{2n}\right)$ as follows

$∆\left(c\_{(1)}, P\_{(1,2)}^{2n}\right)=\pm \left|c\_{\left(1\right)}-c\_{(1, 1)}\right|$*,*

$∆\left( P\_{(3,4)}^{2n}, c\_{(1)}\right)=\pm \left|c\_{\left(4, n\right)}- c\_{\left(1\right)}\right|$

$∆\left( P\_{(1,2)}^{2n}, P\_{(3, 4)}^{2n}\right)=\pm \left|c\_{\left(2, n\right)}- c\_{\left(3, 1\right)}\right|$.

 So, the list of difference of $C\_{v-1}$ shall be represented as a follows:

$∆\left(C\_{v-1}\right)=∆\left( P\_{(1,2)}^{2n}\right)∪∆\left( P\_{(3, 4)}^{2n}\right)∪∆\left(c\_{(1)}, P\_{(1,2)}^{2n}\right)∪∆\left( P\_{(3,4)}^{2n}, c\_{(1)}\right)∪∆\left( P\_{(1,2)}^{2n}, P\_{(3, 4)}^{2n}\right)$.

Now we are able to provide our main result.

Theorem 1.There exists a full cyclic near Hamiltonian cycle system of $4K\_{v}$, $CNHC\left(4K\_{v},C\_{v-1}\right)$, when $v=4n+2, n>2$.

Proof. Suppose $F=\left\{C\_{4n+1}, C\_{4n+1}^{\*} \right\}$ is a set of near Hamiltonian cycles of $4K\_{4n+2}$ where

$C\_{4n+1}=\left(1, P\_{(1,2)}^{2n}, P\_{(3,4)}^{2n}\right)$*,*

$ C\_{4n+1}^{\*}=\left(2n+1, P\_{(1,2)}^{\left(2n\right)^{\*}}, P\_{(3,4)}^{\left(2n\right)^{\*}}\right)$

Such that:

* $P\_{(1,2)}^{2n}=\left[4n+1, 2, 4n, 3, …, 3n+2, n+1\right]$.

 $=\left[\bigcup\_{i=1}^{n}4n+2-i, i+1\right]$

* $ P\_{(3, 4)}^{2n}=\left[n+2, 3n+1, n+3, 3n, …, 2n+1, 2n+2\right]==\left[\bigcup\_{i=1}^{n}n+1+i, 3n+2-i\right]$
* $P\_{\left(1,2\right)}^{\left(2n\right)^{\*}}=\left[4n+1, 2n+2, 4n, 2n+3, …, 3n+2, 3n+1\right]=\left[\bigcup\_{i=1}^{n}4n+2-i, 2n+1+i\right]$.
* $P\_{\left(3,4\right)}^{\left(2n\right)^{\*}}=\left[n+1, n, n+2, n-1, …, 2n, 1\right]$

 $=\left[\bigcup\_{i=1}^{n}n+i, n+1-i\right]$.

We will divide the proof into two parts as follows:

Part 1. In this part will be proved that $F$ satisfies a near-4-factor. We shall calculate the vertex set of $C\_{4n+1} $and$ C\_{4n+1}^{\*}$ such that:

$V\left(C\_{4n+1}\right)=V\left(P\_{(1,2)}^{2n}\right)∪V\left(P\_{(3, 4)}^{2n}\right)∪\left\{1\right\}$*.*

$V\left(C\_{4n+1}^{\*}\right)= V(P\_{(1,2)}^{\left(2n\right)^{\*}})∪V\left(P\_{\left(3, 4\right)}^{\left(2n\right)^{\*}}\right)∪\left\{2n+1\right\}$*.*

$\bigcup\_{i=1}^{n}c\_{\left(1, i\right)}=\left\{4n+2-i, 1\leq i\leq n\right\}$

$=\left\{4n+1, 4n, …, 3n+2\right\}$, (1)

 $\bigcup\_{i=1}^{n}c\_{\left(2, i\right)}=\left\{i+1 , 1\leq i\leq n\right\}=\left\{2, 3, …, n+1\right\}$, (2) $\bigcup\_{i=1}^{n}c\_{\left(3, i\right)} =\left\{n+1+i, 1\leq i\leq n\right\}$

 $=\left\{n+2, n+3, …, 2n+1\right\}$, (3)

$\bigcup\_{i=1}^{n}c\_{\left(4, i\right)}=\left\{3n+2-i , 1\leq i\leq n\right\}$

 $=\left\{3n+1, 3n, …, 2n+2\right\}$. (4)

From above equations, it is easy to notice that $V\left(C\_{4n+1}\right)$ covers each nonzero element of $Z\_{4n+2}$ exactly once.

$\bigcup\_{i=1}^{n}c\_{\left(1, i\right)}^{\*} =\left\{4n+2-i, 1\leq i\leq n\right\}$

 $=\left\{4n+1, 4n, …, 3n+2\right\}$, (5)

$\bigcup\_{i=1}^{n}c\_{\left(2, i\right)}^{\*}=\left\{2n+1+i, 1\leq i\leq n\right\}$

 $=\left\{2n+2, 2n+3, …, 3n+1\right\}$, (6)

$\bigcup\_{i=1}^{n}c\_{\left(3, i\right)}^{\*} =\left\{n+i, 1\leq i\leq n\right\}$

 $=\left\{n+1, n+2, …, 2n\right\}$ (7)

$\bigcup\_{i=1}^{n}c\_{\left(4, i\right)}^{\*}=\left\{n+1-i\right\}=\left\{n, n-1, …, 1\right\}$ (8)

 It can be observed from the above equations that $V\left(C\_{4n+1}\right)= V\left(C\_{4n+1}^{\*}\right)$. Then, the multiset $V\left(C\_{4n+1}\right) ∪ V\left(C\_{4n+1}^{\*}\right)$ covers each nonzero elements of $Z\_{4n+2}$ exactly twice. Since the cycle graph is $2$-regular graph, therefore $F=\left\{C\_{4n+1}, C\_{4n+1}^{\*} \right\}$ satisfies a near-four-factor (with focus zero).

Part 2. In this part we will prove $F=\left\{C\_{4n+1}, C\_{4n+1}^{\*} \right\}$ is the starter set of cyclic $\left(v-1\right)$-cycle system of $4K\_{v}$. So, we will calculate the difference set of each of them as follows:

$∆(C\_{4n+1})= ∆\left(P\_{(1,2)}^{2n}\right)∪∆\left(P\_{\left(3, 4\right)}^{2n}\right)∪∆\left(c\_{\left(1\right)}, P\_{\left(1,2\right)}^{2n}\right)∪ ∆\left( P\_{\left(1,2\right)}^{2n}, P\_{\left(3, 4\right)}^{2n}\right)∪∆\left( P\_{\left(3,4\right)}^{2n}, c\_{\left(1\right)}\right) $ (9)

* + $∆\_{1}\left(P\_{\left(1,2\right)}^{2n}\right)= \bigcup\_{i=1}^{n}\pm \left|4n+1-2i\right|=\left\{4n-1, 4n-3, …, 2n+1\right\}∪\left\{3, 5, …, 2n+1\right\} $.
	+ $∆\_{2}\left(P\_{(1,2)}^{2n}\right)=\bigcup\_{i=1}^{n-1}\pm \left|4n-2i\right|=\left\{4n-2, 4n-4, …, 2n+2\right\}∪\left\{4, 6, …, 2n\right\}$.
	+ $∆\_{1}\left(P\_{\left(3, 4\right)}^{2n}\right)=\bigcup\_{i=1}^{n}\pm \left|2n+1-2i\right|=\left\{2n-1, 2n-3, …, 1\right\}∪\left\{2n+3, 2n+5, …, 4n+1\right\}$.
	+ $∆\_{2}\left(P\_{(3, 4)}^{2n}\right)=\bigcup\_{i=1}^{n-1}\pm \left|2n-2i\right|=\left\{2n-2, 2n-4, …, 2\right\}∪\left\{2n+4, 2n+6, …, 4n\right\}$
	+ $∆\left(c\_{(1)}, P\_{(1,2)}^{2n}\right)=\pm \left|c\_{\left(1\right)}- c\_{\left(1, 1\right)}\right|=\left\{2, 4n\right\}$,
	+ $∆\left( P\_{(1,2)}^{2n}, P\_{(3, 4)}^{2n}\right)=\pm \left|c\_{\left(2, n\right)}-c\_{\left(3, 1\right)}\right|=\left\{1, 4n+1\right\}$.
	+ $∆\left( P\_{(3,4)}^{2n}, c\_{(1)}\right)=\pm \left|c\_{\left(4, n\right)}-c\_{\left(1\right)}\right|=\left\{2n+1, 2n+1\right\}$

From the equation 9, We note that the list of differences of$C\_{4n+1}$*,* $∆\left(C\_{4n+1}\right)$, covers each nonzero elements of $Z\_{4n+2}$ twice except the differences $\left\{2n, 2n+2\right\}$ appear once.

 Now we will calculate $∆\left(C\_{4n+1}^{\*}\right)$ such as

$∆\left(C\_{4n+1}^{\*}\right)=∆\left(P\_{\left(1,2\right)}^{\left(2n\right)^{\*}}\right)∪∆\left(P\_{\left(3,4\right)}^{\left(2n\right)^{\*}}\right)∪∆\left(c\_{\left(1\right)}^{\*}, P\_{\left(1,2\right)}^{\left(2n\right)^{\*}}\right)∪ ∆\left(P\_{\left(1,2\right)}^{\left(2n\right)^{\*}}, P\_{\left(3, 4\right)}^{\left(2n\right)^{\*}}\right)∪∆( P\_{\left(3, 4\right)}^{\left(2n\right)^{\*}}, c\_{\left(1\right)}^{\*})$(10)

* $∆\_{1}\left(P\_{\left(1,2\right)}^{\left(2n\right)^{\*}}\right)=\bigcup\_{i=1}^{n}\pm \left|2n+1-2i\right|=\left\{2n-1, 2n-3, …, 1\right\}∪\left\{2n+3, 2n+5, …, 4n+1\right\}$.
* $∆\_{2}\left(P\_{\left(1,2\right)}^{\left(2n\right)^{\*}}\right)=\bigcup\_{i=1}^{n-1}\pm \left|2n-2i\right|=\left\{2n-2, 2n-4, …, 2\right\} ∪\left\{2n+4, 2n+6, …, 4n\right\}$.
* $∆\_{1}\left(P\_{\left(3, 4\right)}^{\left(2n\right)^{\*}}\right)=\bigcup\_{i=1}^{n}\pm \left|2i-1\right|=\left\{1, 3, …, 2n-1\right\}∪\left\{4n+1, 4n-1, …, 2n+3\right\}$.
* $∆\_{2}\left(P\_{\left(3, 4\right)}^{\left(2n\right)^{\*}}\right)=\bigcup\_{i=1}^{n-1}\pm \left|2i\right|=\left\{2, 4, …, 2n-2\right\} ∪\left\{4n, 4n-2, …, 2n+4\right\}$.
* $∆\left(c\_{\left(1\right)}^{\*}, P\_{\left(1,2\right)}^{\left(2n\right)^{\*}}\right)=\pm \left|c\_{\left(1\right)}^{\*}- c\_{\left(1,1\right)}^{\*}\right|=\left\{2n, 2n+2\right\}$.
* $∆\left(P\_{\left(1,2\right)}^{\left(2n\right)^{\*}}, P\_{\left(3, 4\right)}^{\left(2n\right)^{\*}}\right)=\pm \left|c\_{\left(2, n\right) }^{\*}- c\_{\left(3, 1\right)}^{\*}\right|=\left\{2n, 2n+2\right\}$.
* $∆\left( P\_{\left(3, 4\right)}^{\left(2n\right)^{\*}}, c\_{\left(1\right)}^{\*}\right)=\pm \left| c\_{\left(4, n\right)}^{\*}- c\_{\left(1\right)}^{\*}\right|=\left\{2n, 2n+2\right\}$.

As clearly shown, in the equations 10, every nonzero element in $Z\_{4n+2}$ appears twice except $\left\{2n, 2n+2\right\}$ appear three times in $∆\left(C\_{4n+1}^{\*}\right)$. Based on Lemma 1, the cycles $\left\{C\_{4n+1}, C\_{4n+1}^{\*} \right\}$ have trivial stabilizer.

One can easily note that $∆\left(F\right)=∆\left(C\_{4n+1}\right)∪∆\left(C\_{4n+1}^{\*}\right)$ covers each non zero integers in $Z\_{4n+2}$ four times. Thus, $F=\left\{C\_{4n+1}, C\_{4n+1}^{\*}\right\}$ is the starter cycles of cyclic $\left(v-1\right)$-cycle system of$ 4K\_{v}$ by Lemma 2. Hence, the cycles set $F=\left\{C\_{4n+1}, C\_{4n+1}^{\*}\right\}$ generates a full near Hamiltonian cycle system of $4K\_{v}$ by adding one modular $v$ when $v=4n+2, n>2$ □

# cyclic Hamiltonian wheel system

A wheel graph of order $m$, denoted by $W\_{m}$, consists of a singleton graph $K\_{1}$ and a cycle graph of order $m-1$, $C\_{m-1}$, in which the $K\_{1}$ is connected to all the vertices of $C\_{m-1}$, written $K\_{1}+C\_{m-1}$ or $c\_{0}+\left(c\_{1}, c\_{2}, …, c\_{m-1}\right)$. An $m$-wheel contains $2\left(m-1\right)$ edges such that the edge set of $W\_{m}$ is $E\left(W\_{m}\right)=E\left(K\_{1, \left(m-1\right)}\right)∪E\left(C\_{m-1}\right)$ [12].

An $m$-wheel system of graph $G$ is a decomposition of edge set of $G$ into collection $W=\left\{W\_{m\_{1}}, …, W\_{m\_{r}}\right\}$ of edges- disjoint of $m$-wheels. Similar to the cyclic cycle system, an $m$-wheel system of $λK\_{v}$ is a cyclic if $V\left(λK\_{v}\right)=Z\_{v}$ and if $W\_{m}=c\_{0}+\left(c\_{1}, c\_{2}, …, c\_{m-1}\right)\in W$ implies that $W\_{m}+1=\left(c\_{0}+1\right)+\left(c\_{1}+1, c\_{2}+1, …, c\_{m-1}+1\right)$ is also in $W$. Moreover, if $m=v$ then it is called a cyclic Hamiltonian wheel system. The list of difference of $W\_{m}=c\_{0}+\left(c\_{1}, c\_{2}, …, c\_{m-1}\right)$ is $Δ\left(W\_{m}\right)=Δ\left(K\_{1, \left(m-1\right)}\right)∪Δ\left(C\_{m-1}\right)$ such that $Δ\left(C\_{m}\right)=\left\{\pm \left|c\_{i+1}- c\_{i}\right|ǀ 1\leq i\leq m-1\right\}$ where $c\_{m}=c\_{0}$ and $Δ\left( K\_{\left(1,\left(m-1\right)\right)}\right)=\left\{\pm \left|c\_{i}- c\_{0}\right| ǀ 1\leq i\leq m-1\right\}$. More generally, given a multiset $W=\left\{W\_{m\_{1}}, …, W\_{m\_{t}}\right\}$ of $m$-wheels of $λK\_{v}$, the list of differences from $W$ is $∆W=\bigcup\_{i=1}^{t}∆W\_{m\_{i}}$ .

Definition 2. The full cyclic Hamiltonian wheel system of a graph$ 8K\_{v}$, denoted by$ CHWS(8K\_{v},W)$, is a cyclic $\left(v\right)$-wheel system of a graph$ 8K\_{v}$ that generated by starter set $W=\left\{W\_{v\_{1}}, …, W\_{v\_{t}}\right\}$ such that the associated cycles with wheels satisfy near-four-factor with focus singleton graph.

In other words $CHWS(8K\_{v}, W)$ is a $\left(v×\left|W\right|\right)$ array such that satisfies the following conditions:

1. The wheels in row $r$ form a $r+$(near-four-factor).
2. The wheels associated with the rows contain no repetitions.

For clarity, we provide an example to demonstrate the construction of a cyclic Hamiltonian wheel system stated above.

Example 2. Let $G=8K\_{14}$ and $W=\left\{ W\_{14}, W\_{14}^{\*}\right\}$ is a set of Hamiltonian wheels of $8K\_{14}$. Where$ W\_{14}=K\_{1}+C\_{13}=0+\left(1, 13, 2, 12, 3, 11, 4, 5, 10, 6, 9, 7, 8\right)$ and $W\_{14}^{\*}=K\_{1}^{\*}+C\_{13}^{\*}=0+\left(13, 8, 12, 9, 11, 10, 4, 3, 5, 2, 6, 1, 7\right)$. From the Example $1$, we can note that the $13$-cycles $\left\{C\_{13}, C\_{13}^{\*}\right\}$ satisfy a near-four-factor with focus 0 (zero element). Moreover, the list of differences from $\left\{C\_{13}, C\_{13}^{\*}\right\}$ covers every nonzero element of $Z\_{14}$ exactly four times. Now we want to find the list of differences from $\left(K\_{1, \left(13\right)}\right)∪\left(K\_{1, \left(13\right)}^{\*}\right)$ as a follows

$$∆\left(K\_{1, \left(13\right)}\right)=\left\{\pm \left|c\_{i}- c\_{0}\right| ǀ 1\leq i\leq 13\right\}$$

 $=\left\{\pm \left|c\_{i}\right| ǀ 1\leq i\leq 13\right\}$

 such that the $c\_{i} ϵV\left(C\_{13}\right), 1\leq i\leq 13$. Since the vertex set of $C\_{13} $is $Z\_{14}-\left\{0\right\}$ and $Z\_{14}=- Z\_{14}$, then $∆\left(K\_{1, \left(13\right)}\right)$ covers each nonzero element of $Z\_{14}$ twice. Similarly, one can be found $∆\left(K\_{1, \left(13\right)}^{\*}\right)=∆\left(K\_{1, \left(13\right)}\right)$. So $∆\left(W\right)$ covers each nonzero of $Z\_{14}$ four times. Thus, $W=\left\{ W\_{14}, W\_{14}^{\*}\right\}$ is the starter set of $CHWS(8K\_{14}, W)$.

Then $CHWS(8K\_{14}, W\_{14})$ is an $\left(14×2\right)$ array deign where all its wheels can be generated by repeated addition 1 (modular $14$) on the starter set $W$ as shown in the Table III.

TABLE III

 $CHWS(8K\_{14}, W\_{14})$

|  |
| --- |
| $$CHWS\left(8K\_{14}, W\right)$$ |
| $$0+\left(1,13,2,12,3,11,4,5,10,6,9,7,8\right)$$ | $$0+\left(13,8,12,9,11,10,4,3,5,2,6,1,7\right)$$ |
| $$1+\left(2,0,3,13,4,12,5,6,11,7,10,8,9\right)$$ | $$1+\left(0,9,13,10,12,11,5,4,6,3,7,2,8\right)$$ |
| $$2+\left(3,1,4,0,5,13,6,7,12,8,11,9,10\right)$$ | $$2+\left(1,10,0,11,13,12,6,5,7,4,8,3,9\right)$$ |
| $$\vdots $$ | $$\vdots $$ |
| $$13+\left(0,12,1,1,2,10,3,4,9,5,8,6,7\right)$$ | $$13+\left(12,7,11,8,10,9,3,2,4,1,5,0,6\right)$$ |

The following theorem proves the existence of $CHWS\left(8K\_{4n+2}, W\right)$.

Theorem 2. There exists a full cyclic Hamiltonian wheel system of $8K\_{v}$, $CHWS\left(8K\_{v}, W\right)$, for $v=4n+2, n>2$.

Proof. We have to present a starter set $W=\left\{K\_{1}+C\_{4n+1}, K\_{1}^{\*}+C\_{4n+1}^{\*}\right\}$ of $CHWS\left(8K\_{v}, W\right)$ such that the cycles associated with the wheels in $W$ satisfy a near-four-factor with focus a singleton graph.

Suppose $W=\left\{0+C\_{4n+1}, 0+C\_{4n+1}^{\*} \right\}$ is a set of Hamiltonian wheels of $8K\_{4n+2}$ where

 $C\_{4n+1}=\left(1, P\_{(1,2)}^{2n}, P\_{(3,4)}^{2n}\right)$ ,

 $ C\_{4n+1}^{\*}=\left(2n+1, P\_{(1,2)}^{\left(2n\right)^{\*}}, P\_{(3,4)}^{\left(2n\right)^{\*}}\right)$

Such that:

* $P\_{(1,2)}^{2n}=\left[4n+1, 2, 4n, 3, …, 3n+2, n+1\right]$.
* $ P\_{(3, 4)}^{2n}=\left[n+2, 3n+1, n+3, 3n, …, 2n+1, 2n+2\right]$.
* $P\_{\left(1,2\right)}^{\left(2n\right)^{\*}}=\left[4n+1, 2n+2, 4n, 2n+3, …, 3n+2, 3n+1\right]$.
* $P\_{(3,4)}^{\left(2n\right)^{\*}}=\left[n+1, n, n+2, n-1, …, 2n, 1\right]$.

From Theorem 1, the cycles associated with the Hamiltonian wheels in $W$ satisfy the near-four-factor with focus zero element.

 Now, we want to prove $W=\left\{K\_{1}+C\_{4n+1}, K\_{1}^{\*}+C\_{4n+1}^{\*}\right\}$ is a $\left(λK\_{v}, W\right)$-difference system. To do this, it is enough to show that the list of differences

$∆W=\left\{∆\left(C\_{4n+1}\right)∪∆\left(C\_{4n+1}^{\*}\right)∪∆\left(K\_{1, \left(4n+1\right)}^{\*}\right)∪ ∆\left(K\_{1, \left(4n+1\right)}\right)\right\}$

covers each element of $\left\{Z\_{4n+2}-\left\{0\right\}\right\}$ eight times. Firstly, as indicated in Theorem 1, the list of differences of $\left\{\left(C\_{4n+1}\right)∪\left(C\_{4n+1}^{\*}\right)\right\}$ cover each nonzero element in $Z\_{4n+2}$ exactly four times.

Secondly, the list of differences of $\left(K\_{1, \left(4n+1\right)}\right)$ is $\left\{\pm \left|c\_{i}- 0\right| ǀ c\_{i}\in C\_{4n+1}\right\}$. Since $V\left(C\_{4n+1}\right)=Z\_{4n+2}-\left\{0\right\}$ then $\left\{\left|c\_{i}- 0\right| ǀ c\_{i}\in C\_{4n+1}\right\}= Z\_{4n+2}-\left\{0\right\}$. Because of $Z\_{4n+2}=-\left\{Z\_{4n+2}\right\}$ , then $∆\left(K\_{1, \left(4n+1\right)}\right)=\left\{\pm \left|c\_{i}- 0\right| ǀ c\_{i}\in C\_{4n+1}\right\}$ covers each nonzero element of $Z\_{4n+2}$ twice. Likewise, we repeat the same strategy on cycle $K\_{1, \left(4n+1\right)}^{\*}$ to find $∆\left(K\_{1, \left(4n+1\right)}^{\*}\right)$. Also, it is an easy matter to check that $∆\left(K\_{1, \left(4n+1\right)}^{\*}\right)=∆\left(K\_{1, \left(4n+1\right)}\right)$.

Linking together the above list of differences, we see that $∆W$ covers each nonzero element of $Z\_{4n+2}$ eight times. On the other hand, each wheel graph in $W$ has trivial stabilizer based on Lemma 1. Therefore, $W$ is the starter set of $CHWS(8K\_{v},W)$, by Lemma 2. One can be generated $CHWS(8K\_{v},W)$ by repeated addition 1 modular $v$ on $W$. □

# Conclusion

In this paper, we have provided new designs $CNHC\left(4K\_{v}, C\_{v-1}\right)$ and $CHWS\left(8K\_{v}, W\_{v}\right)$ where $v≡2\left(mod4\right)$. These designs are interested in a decomposition of complete multigraph into cyclic $\left(v-1\right)$-cycle and cyclic $\left(v\right)$-wheel graphs, respectively. We have also proved the existence of these designs by constructed the starter set for each of them. Moreover, one can ask if $CNHC\left(2λK\_{v}, C\_{v-1}\right)$ and $CHWS\left(2λK\_{v}, W\_{v}\right)$ can be constructed for the case $v≡2, 4\left(mod 4\right)$ and $λ>2$.

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